POLYA TYPE DISTRIBUTIONS IN RENEWAL THEORY WITH AN APPLICATION TO AN INVENTORY PROBLEM

BY

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1. INTRODUCTION AND SUMMARY

This thesis has two purposes: (a) to solve an important practical problem arising in inventory theory, (b) to present some new results on Polya Type distributions arising in renewal theory. Since the derivation of the solution of the inventory problem led to the results obtained in renewal theory, it will be motivating to present the inventory problem first:

1.1 Inventory Problem  A complex system is to be placed in the field for a period of time $t_0$. When a component fails, it is instantly replaced by a spare, if available. The components considered operate independently (i.e., failure of one does not influence failure of any other), and are essential to continued system operation, so that a shortage of any component considered results in system shutdown. Only the spares originally provided may be used for replacements; i.e., no resupply of spares can occur during the period.

The system contains $d_i$ components of type $i$, $i = 1, 2, \ldots, k$, simultaneously operating. Different components of a given type may be used at different levels of intensity, so that the length of life of the $j^{th}$ component of type $i$ (component $i, j$) and its replacements are independent random variables with density $f_{ij}(t)$, $j = 1, \ldots, d_i; i = 1, \ldots, k$. The scheduled length of time $t_{ij}$ that component $i, j$ and its replacements must operate during $t_0$ is a constant, $j = 1, \ldots, d_i; i = 1, \ldots, k$. Finally, the cost of a single unit of type $i$ is $c_i$, $i = 1, \ldots, k$. 
What choice of \( n_i \), the number of spares of type \( i \) initially provided, \( i = 1, \ldots, k \), will yield maximum assurance of continued system operation (i.e., no shortage of any essential component) during the period \([0, t_0]\) at a total cost for spares

\[
c(n_1, \ldots, n_k) = \sum_{i=1}^{k} n_i c_i \leq c_0, \text{ a constant?}
\]

1.2 Related Models Related models in the spare parts problem have been treated in (3), (4), and (5). In these models, the expected total of weighted shortages is minimized subject to a linear weight or cost restraint, with the demand probability density for spares assumed a priori. In our model, we maximize assurance of continued system operation by optimal allocation of spares likewise subject to a linear restraint, but with the demand for spares, instead of being assumed a priori, generated by failure of operating units following known probability distributions. Thus, to obtain the composition of the optimal spare parts kit we use information about component failure distributions rather than information about component demand distributions. In the typical situation under consideration—a new system under experimentation for a single period in the field—we are thus given the opportunity to use information we may have, component failure rates, rather than called upon to provide information we may not have, component demand distributions. Our choice of probability of continued system operation as the figure of merit to be maximized is especially relevant in military applications, where a penalty cost is often difficult to determine.
1.3 Sketch of Results In Section 2, we present a procedure for obtaining the maximizing set $n_1^*, \ldots, n_k^*$ for the case $Q_i(n + 1)/Q_i(n)$ a decreasing function of $n$, where $Q_i(n)$ is the probability that $n$ or less failures of type $i$ occur during $[0,t_0]$, $i = 1, \ldots, k$. In Section 3, we show that $Q_i(n + 1)/Q_i(n)$ is a decreasing function of $n$ whenever each $f_{ij}(t)$ for $j = 1, \ldots, d_i$ is a density having a monotone likelihood ratio in differences of $t$; since the class of monotone likelihood ratio densities includes most of the standard probability densities, the Inventory Problem above may be considered solved.

In order to demonstrate that $f_{ij}(t)$, a monotone likelihood ratio density for each $j = 1, \ldots, d_i$ implies $Q_i(n + 1)/Q_i(n)$ a decreasing function of $n$, we derive some results of independent interest in renewal theory, of the following type (See Sections 3, 4):

**Theorem** If $f_i(t)$ is a Polya frequency function of order $k$ having a continuous first derivative, with $f_i(t) = 0$ for $t < 0$, $i = 1, 2, \ldots$, then $p'(n,t)$, the convolution of $f_1$ with $f_2$ with $\ldots$ with $f_n$, is a Polya type density of order $k$ in $n$ and $t$, where $n$ ranges over $1, 2, \ldots$.

**Theorem** If $f_i(t)$ are monotone likelihood ratio densities in differences of $t$ with $f_i(t) = 0$ for $t < 0$, $i = 1, 2, \ldots$, then $\Delta p'(n,t) = \int_t^\infty p'(n + 1, u)du - \int_t^\infty p'(n,u)du$ has the monotone likelihood ratio property in $n$ and $t$ jointly, and also in differences of $n$. $(\Delta p'(n,t)$ is the probability that precisely $n$ events occur during $[0,t]$ if the interval between the $i-1$st and $i$th
event is an independent observation from $f_i$, $i = 1, 2, \ldots$.) We then apply these theorems to the Inventory Problem and show how to solve more general models in which the $t_{ij}$ are random variables, the original components are aged, and in which the replacements may have different failure densities.

In Section 5 we show how to compute with explicit formulas the composition of the optimal spare parts kit for the most commonly occurring real life situation, namely where component failure densities are exponential. A numerical example is presented to illustrate the method. Finally, we point out how the mathematical model may be applied in achieving maximum reliability in the design of complex systems.

In Section 6, we consider the extension of the theorems obtained for positive random variables to the more general case in which the random variables may be positive or negative. The corresponding conclusions are considerably weaker in the latter case.

2. MAXIMIZING A NONLINEAR FUNCTION SUBJECT TO A LINEAR CONSTRAINT

2.1 Maximizing $Q(n)$ As A Function Of The $Q_i(n)$

Let $Q_i(n) = \text{probability that } n_i \text{ or less failures of type } i \text{ occur during } [0, t_0], \ i = 1, \ldots, k.$

$n = n_1, n_2, \ldots, n_k,$ a vector.

$Q(n) = \text{probability that no shortage is experienced for any of the } k \text{ types during } [0, t_0] \text{ given an initial kit of composition } n.$

$c(n) = \text{total cost of a spare parts kit composed of } n_i \text{ units of type } i, \ i = 1, 2, \ldots, k.$
Since components operate independently and failure of any one causes system shutdown, the probability of continued system operation during \([0, t_0]\) is given by

\[
Q(n) = \prod_{i=1}^{k} Q_i(n_i)
\]  
(2.1)

Also, we note that

\[
c(n) = \sum_{i=1}^{k} n_i c_i
\]  
(2.2)

We wish to maximize \(Q(n)\) subject to

\[
\sum_{i=1}^{k} n_i c_i \leq c_o \text{ and } n_i \geq 0, \; i = 1, \ldots, k.
\]  
(2.3)

Define: \(R_i(n) = \ln Q_i(n)\) and \(R(n) = \ln Q(n)\). Then, it is equivalent to maximize \(R(n)\) subject to (2.3).

Maximizing a nonlinear function \(R(n)\) subject to linear restraints (2.3) is a special case of nonlinear programming, treated in (10). In (10), the theorems are developed in detail for continuous variables. In our problem, we are dealing with discrete variables, \(n_1, \ldots, n_k\). Thus we shall independently derive the required theorems.

Define \(\Delta R_i(n) = R_i(n+1) - R_i(n)\) for \(n = 0, 1, 2, \ldots; i = 1, 2, \ldots, k\). Then we shall show in Theorem 2.1 and its Corollary below that when \(\Delta R_i(n)\) decreasing for \(i = 1, \ldots, k\) we may obtain the solution to our Inventory Problem by following Procedure 1.

Procedure 1 Let \(\Delta R_i(n)\) be a decreasing function of \(n\) for \(i = 1, \ldots, k\). For arbitrary \(r > 0\), for those \(i\) such that
\( \Delta R_i(0) < r c_i \), define \( n_i^*(r) = 0 \); for the remaining \( i \), define \( n_i^*(r) \) as \( 1 + \) largest \( n \) such that \( \Delta R_i(n) \geq r c_i \). Compute

\[
c(n^*(r)) = \sum_{i=1}^{k} c_i n_i^*(r).
\]

Let \( r_o \) be the value of \( r \) yielding maximum \( c(n^*(r)) \leq c_o \). Then \( n^*(r_o) \) is the spares kit corresponding to a budget restraint of \( c_o \).

The following theorem shows \( n^* \) is optimal when \( c_o \) is one of the values assumed by \( c(n^*(r)) \) as \( r \) varies over \( (0, \infty) \).

**Theorem 2.1** \( n^* \) maximizes \( R(n) \) among all \( n \) such that \( c(n) \leq c(n^*) \), \( n \geq 0 \).

**Proof** We will show for any \( 0 \leq n \equiv n^* \) for which \( c(n) \leq c(n^*) \) that \( R(n) \leq R(n^*) \). Suppose \( n_i > n_i^* \) for \( i \) in \( I_1 \), \( n_i < n_i^* \) for \( i \) in \( I_2 \), where \( I_1, I_2 \) are subsets of \( \{ 1, 2, \ldots, k \} \).

Since \( \Delta R_i(n_i) \) is a decreasing function of \( n_i \) by assumption, then

\[
\Delta R_i(n_i^* + j) < r c_i \quad \text{for} \quad i \in I_1, \quad j = 1, 2, \ldots, n_i - n_i^*.
\]

(2.4a)

Similarly, for \( i \) in \( I_2 \),

\[
\Delta R_i(n_i^* - j) \geq r c_i \quad \text{for} \quad i \in I_2, \quad j = 1, 2, \ldots, n_i^* - n_i.
\]

(2.4b)

Hence

\[
R(n) - R(n^*) = \sum_{i \in I_1} \sum_{j=1}^{n_i^*-n_i} \Delta R_i(n_i^*+j) - \sum_{i \in I_2} \sum_{j=1}^{n_i^*-n_i} \Delta R_i(n_i^*-j)
\]

\[
\leq r \sum_{i \in I_1} (n_i^*-n_i)c_i - r \sum_{i \in I_2} (n_i^*-n_i)c_i = r \sum_{i=1}^{k} (n_i^*-n_i)c_i
\]

\[= r \{ c(n) - c(n^*) \} . \]
But \( r > 0 \) and \( c(n) - c(n^*) \leq 0 \). Hence \( R(n) \leq R(n^*) \).

QED

(In the remainder of this section, we shall confine attention to the case in which \( AR_1(n) \) is a decreasing function of \( n \) for \( i = 1, \ldots, k \). In Section 3, we derive a sufficient condition for \( AR_1(n) \) decreasing.)

It follows that a given value of \( r \) yields a kit composition \( n^*_i(r) \) which solves the original inventory problem for the particular case \( c_o = \sum_{i=1}^{k} c_i n_i^*(r) \). Hence by varying \( r \) over the interval \((0, \infty)\), we may obtain solutions corresponding to a \( c_o \) equal to any of the different (discrete) values \( \sum_{i=1}^{k} c_i n_i^*(r) \) may assume. We shall refer to any vector \( n^*_i \) obtained by Procedure 1 as an optimal point or an optimal kit.

For values of \( c_o \) different from any of the \( c(n^*) \), Procedure 1 may provide only an approximate solution because of the discrete nature of the variables \( n_1, \ldots, n_k \); the bound on the error is given in the Corollary below, and in most practical problems will be small.

**Corollary** For \( c_o \neq c(n^*_i(r)) \) for all \( r > 0 \),

\[
\max_{c(n) \leq c_o, n > 0} Q(n) - Q(n^*_i(r_o)) \leq Q(n') - Q(n^*_i(r_o))
\]

where \( c(n') = \min_{c(n^*_i) > c_o} c(n^*_i) \). (\( n' \) represents the cheapest optimal kit of higher cost than \( c_o \).)

**Proof** By Th. 2.1, \( R(n') \geq R(n) \) for all \( n > 0 \) such that \( c(n) \leq c_o < c(n') \). Hence \( Q(n') \geq Q(n) \) for all \( n > 0 \) such that
Thus the error in using Procedure 1 is at most the difference in protection achieved by the two adjacent optimal kits whose costs straddle the specified $c_o$. This motivates the following alternate procedure for obtaining in succession the same set of optimal points as are obtained by Procedure 1.

Procedure 2  First note that $(0, \ldots, 0)$ is an optimal point (corresponding to $c_o = 0$). Next let $\hat{n}^*$ be any optimal point. Compute $\Delta R_i(n_i^*)/c_i$ for $i = 1, \ldots, k$. If the $\alpha^{th}$ ratio is the largest, the next optimal point to the right of $\hat{n}^*$ is $(n_1^*, \ldots, n_{\alpha-1}^*, n_{\alpha}^* + 1, n_{\alpha+1}^*, \ldots, n_k^*)$, as will be proved in Theorem 2.2. (We shall designate the immediate successor of an optimal point $\hat{n}^*$ by $\hat{n}^*(\alpha)$, where $\alpha$ is the (only) coordinate that has changed.)

Theorem 2.2 Any point obtainable by Procedure 1 is obtainable by Procedure 2, and conversely.

Proof  Let $\hat{n}^*$ be a point obtainable by Procedure 2. Suppose $\hat{n}^*$ is obtained from the preceding point on the left by a change in the $\alpha^{th}$ coordinate. Set $r_o = \Delta R_\alpha (n_\alpha^* - 1)/c_\alpha$. Thus

$$\Delta R_\alpha(n_\alpha^*)/c_\alpha < r_o \leq \Delta R_\alpha(n_\alpha^* - 1)/c_\alpha$$

since $\Delta R_\alpha(n)$ is decreasing. Also we know

$$\Delta R_i(n_i^*)/c_i < r_o$$

for all $i \neq \alpha$. (2.6)

Now if there existed $\beta$ such that

$$\Delta R_\beta(n_\beta^* - 1)/c_\beta < r_o \leq \Delta R_\alpha(n_\alpha^* - 1)/c_\alpha$$

there would be a point lying to the left of $\hat{n}^*$ whose $\alpha^{th}$ coordinate would be $n_\alpha^*$ and
whose $\beta^{th}$ coordinate would be $\leq n_\beta^* - 1$ (i.e., in computing successive points, the $\alpha^{th}$ coordinate would change before the $\beta^{th}$).

This would contradict the assumption that $n^*$ is obtained from the preceding point on the left by a change in the $\alpha^{th}$ coordinate.

Thus we conclude that

$$\Delta R_i (n_i^* - 1)/c_i \geq r_o \text{ for all } i \neq \alpha.$$  \hspace{1cm} (2.7)

From (2.5), (2.6), and (2.7), we see that $n^*$ is obtainable by Procedure 1.

Next, let $n^*$ and $n^*(\alpha)$ be a pair of successive points obtained by using Procedure 2. We shall show that there are no points lying between them obtainable by Procedure 1. Assume there is such a point $m^*$ with $r = r_o$. For at least one coordinate of $m^*$ other than the $\alpha^{th}$, say the $\beta^{th}$, we thus have either (1) $m^*_\beta > n^*_\beta$, or (2) $m^*_\beta \leq n^*_\beta$. Assume (1). Then the $\beta^{th}$ coordinate of $n^*$, $m^*$, and $n^*(\alpha)$ are, respectively, $n^*_\beta$, $m^*_\beta$, and $n^*_\beta$. Since $n^*$ and $n^*(\alpha)$ are obtainable by Procedure 2, we have just shown that each is obtainable by Procedure 1. Hence there is an $r = r_1$ and an $r = r_2$ with $r_1 > r_2$ associated with $n^*$ and $n^*(\alpha)$ respectively with the property that $\Delta R_\beta (n^*_\beta - 1)/c_\beta \geq r_1$ and $\Delta R_\beta (n^*_\beta )/c_\beta < r_2$. Also $\Delta R_\beta (m^*_\beta )/c_\beta \geq r_o$, while $\Delta R_\beta (m^*_\beta )/c_\beta < r_o$. But $m^*$ lies to the left of $n^*(\alpha)$, so that $r_o > r_2$; hence we have $\Delta R_\beta (n^*_\beta )/c_\beta < r_2 < r_o \leq \Delta R_\beta (m^*_\beta - 1)/c_\beta$. This is impossible since $n^*_\beta \leq m^*_\beta - 1$.

A similar argument rules out (2). Thus there are no points obtainable by Procedure 1 lying between $n^*$ and $n^*(\alpha)$. This means that every point obtainable by Procedure 1 is obtainable by Procedure 2.

QED

- 9 -
Procedure 2 represents a systematic way of obtaining all optimal points needed, and is more convenient for machine computation. Actually the two methods may be usefully combined in the following commonly occurring situation. Suppose there is no single value of $c_0$ specified. Rather, suppose what is desired is a curve (optimal curve) showing maximum protection $Q(n^*)$ against system shutdown as a function of total cost $c(n^*)$ for spares. To obtain this curve, compute an optimal point by Procedure 1 near the lower end of the range of interest of protection or cost (several trial values of $r$ may have to be used). Then proceed systematically up the curve by Procedure 2. The following theorem furnishes some helpful information about the number of points in any portion of the optimal curve.

**Theorem 2.3** The number of points on the optimal curve between any two distinct optimal points $m^*$ and $n^*$ (with $m^* \leq n^*$, say) is

$$\sum_{i=1}^{k} (n_i^* - m_i^*) - 1.$$ 

**Proof** By Procedure 2 we get successive points by changing one coordinate at a time. The number of such changes in going from $m^*$ to $n^*$ is $\sum_{i=1}^{k} (n_i^* - m_i^*)$. Excluding the last change which results in $n^*$, we find that there are $\sum_{i=1}^{k} (n_i^* - m_i^*) - 1$ optimal points between $m^*$ and $n^*$.

QED

2.2 Expressing $Q_i(n)$ As A Function Of The $f_{ij}$ Thus far, we have shown how the problem may be solved in terms of $Q_i(n)$,
\( i = 1, \ldots, k \), under the assumption \( Q_i(n + 1)/Q_i(n) \) decreasing for each \( i \). In the remainder of this section we shall obtain the relationship between the \( Q_i(n) \) and the underlying failure densities \( f_{ij}(t) \). Clearly the random process describing successive replacement of failed components is a renewal process; thus the continued operation of the complex system may be thought of as a set of independent renewal processes.

For \( j = 1, \ldots, d_i; i = 1, \ldots, k; -\infty < t < \infty \), let

\[
p_{ij}(1, t) = f_{ij}(t)
\]

\[
p_{ij}(2, t) = \int_{-\infty}^{\infty} f_{ij}(\theta) f_{ij}(t - \theta) \, d\theta
\]

\[
\vdots
\]

\[
p_{ij}(n, t) = \int_{-\infty}^{\infty} f_{ij}(\theta) p_{ij}(n - 1, t - \theta) \, d\theta \quad \text{for } n = 3, 4, \ldots
\]

\[
p_{ij}(n, t) = \int_{t}^{\infty} p_{ij}(n, \theta) \, d\theta \quad \text{for } n = 1, 2, \ldots
\]

\[
\Delta p_{ij}(n, t) = \begin{cases} 
p_{ij}(1, t) & \text{for } n = 0 \\
p_{ij}(n + 1, t) - p_{ij}(n, t) & \text{for } n = 1, 2, \ldots \end{cases}
\]

\( q_i(n) \) = probability of exactly \( n \) failures of component type \( i \).

Thus \( p_{ij}(n, t) \), being the \( n \)-fold convolution of \( f_{ij}(t) \), represents the probability density for the combined life of the original \( i, j \) component and \( n - 1 \) replacements; as a consequence, \( p_{ij}(n, t_{ij}) \) is the probability that \( n - 1 \) or less spares will be used in replacing the \( i, j \) component during the specified \( t_{ij} \) units of time.

From the definition of \( \Delta p_{ij}(n, t_{ij}) \), we know from renewal theory (1, p.272) that \( \Delta p_{ij}(n, t_{ij}) \) represents the probability that exactly
n spares will be required during the \( t_{ij} \) units of time scheduled for component \( i,j \) (and its replacements). Next, since the total number of spares of type \( i \) required for continued system operation during \([0,t_o]\) is simply the sum of the numbers required to replace component \( i,j \) for \( j = 1, 2, \ldots, d_i \), we have

\[
q_i(n) = \Delta p_{i1}(n,t_{i1}) \ast \Delta p_{i2}(n,t_{i2}) \ast \ldots \ast \Delta p_{id_i}(n,t_{id_i}),
\]

(2.8)

where the convolution (indicated by \( \ast \)) is taken in the first argument of the \( \Delta p_{ij}(n,t_{ij}) \). Finally, \( Q_i(n) \), the probability of \( n \) or less failures of type \( i \) during \([0,t_o]\), is clearly given by

\[
Q_i(n) = \sum_{m=0}^{n} q_i(m),
\]

(2.9)

so that \( q_i(n) \) represents the density and \( Q_i(n) \) the distribution function for the total number of failures (or equivalently, the total number of spares required) of type \( i \) during \([0,t_o]\).

Thus following the definitions above and (2.8) and (2.9), we may compute according to standard procedures (possibly somewhat tediously) the values of \( Q_i(n) \) as a function of the \( f_{ij} \). From the values of \( Q_i(n) \), using either Procedure 1 or Procedure 2 (or a combination), we may solve the original Inventory Problem provided \( Q_i(n + 1)/Q_i(n) \) is decreasing. In the next section, we shall demonstrate that \( Q_i(n + 1)/Q_i(n) \) decreases whenever the \( f_{ij}(t) \) are monotone likelihood ratio densities in differences of \( t \).

3. POLYA TYPE DISTRIBUTIONS IN RENEWAL THEORY

3.1 Preliminaries According to (8, p.282) a family of distributions
\begin{equation*}
F(x, \theta) = \beta (\theta) \int_{-\infty}^{x} f(t, \theta) \, du(t)
\end{equation*}

of a real random variable \( X \) depending on a real parameter \( \theta \) is said to be Polya type \( k \) \((F(x, \theta) \text{ is } PT_k)\) if

\[
0 \leq \begin{vmatrix}
    f(x_1, \theta_1) & \ldots & f(x_1, \theta_m) \\
    \vdots & \ddots & \vdots \\
    f(x_m, \theta_1) & \ldots & f(x_m, \theta_m)
\end{vmatrix}
\]

(symbolized hereafter by \(|f(x, \theta)|\))

for every \( 1 \leq m \leq k \) and all \( x_1 < x_2 < \ldots < x_m \), and \( \theta_1 < \theta_2 < \ldots < \theta_m \). If the family of distributions \( F(x, \theta) \) is \( PT_k \) for every \( k \), then we say that the family is \( PT_\infty \). We shall say that \( f(x, \theta) \) is \( PT_k(\infty) \) if \( F(x, \theta) \) belongs to \( PT_k(\infty) \).

For \( k = 1, 2 \) the condition of being \( PT_k \) reduce to familiar ones. \( f \) is \( PT_1 \) if and only if \( f(x, \theta) \geq 0 \) for all \( x \) and \( \theta \). \( f \) is \( PT_2 \) if and only if it has a monotone likelihood ratio.

Karlin uses the variation diminishing properties of Polya type functions to obtain some fundamental results in statistical decision theory.

We shall find it useful to include in the class of \( PT_k \), functions \( P(\eta, t) = \int_{-\infty}^{\infty} p(\eta, u) \, du, \) where \( p(\eta, u) \) is a probability density and \( \eta \) a real valued parameter, if \(|P(\eta_1, t)| \geq 0 \) for all \( \eta_1 < \ldots < \eta_m \) and \( t_1 < \ldots < t_m \), \( 1 \leq m \leq k \).

If a \( PT_k \) function \( F(x, \theta) \) is a function \( G(x - \theta) \) of the difference of \( x \) and \( \theta \), we shall refer to it as a \textbf{Polya frequency function} of order \( k \) \((G \text{ is } P_f_k)\).
An important property of Polya type functions as defined in (6) is that the convolution of a $PT_m$ function and a $PT_n$ function yields a $PT_{\min(m,n)}$ function (9, lemma 5). We note that this property holds for our slightly more inclusive class of PT functions (assuming the integral involved is finite):

**Lemma 3.1** Let $f(x,\theta)$ be $PT_m\cdot g(\theta,\omega)$ be $PT_n$, and let $h(x,\omega) = \int f(x,\theta) g(\theta,\omega) d\theta < \infty$. Then $h(x,\omega)$ is $PT_{\min(m,n)}$.

**Proof** The proof is as in (9).

Let $p(n,t)$ be the n-fold convolution of the density $f(t)$ for $n = 1, 2, \ldots$; i.e., $p(1,t) = f(t)$; $p(2,t) = \int f(t-\theta) f(\theta) d\theta$; $\ldots$; $p(n+1,t) = \int p(n,t-\theta) f(\theta) d\theta$; $\ldots$. Then, we have immediately:

**Lemma 3.2** If $f(t)$ is Pff$_k$, then $p(n,t)$ is Pff$_k$ in t for $n = 1, 2, \ldots$.

**Proof** This follows from repeated application of Lemma 3.1.

Next, define $H(t) = \begin{cases} 1 & \text{if } t \leq 0 \\ 0 & \text{if } t > 0 \end{cases}$. Then we may show:

**Lemma 3.3** $H(t)$ is Pff$_\infty$.

**Proof** The proof is as in (12, p.335). For any $k$, with $t_1 < \ldots < t_k$, $\theta_1 < \ldots < \theta_k$, $D_k = \mid H(t_j - \theta_j) \mid$ of order $k$, has these properties.

(1) The elements are 0 or 1.

(2) The elements in every row are non-decreasing.

(3) The elements in every column are non-increasing.
From this it follows that $D_k = 0$ or $D_k$ is of the form
\[
\begin{array}{cccccc}
1 & 1 & 1 & \ldots & 1 \\
0 & 1 & 1 & \ldots & 1 \\
0 & 0 & 1 & \ldots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1 \\
\end{array}
= 1
\]
QED

Next define $P(n,t) = \begin{cases} 
0 & \text{for } n < 0 \\
H(t) & \text{for } n = 0 \\
\int_0^\infty p(n,u) \, du & \text{for } n \geq 1.
\end{cases}

Then, we obtain immediately:

Lemma 3.1 If $f(t)$ is Pff$_k$, then $P(n,t)$ is Pff$_k$ in $t$ for $n = 1, 2, \ldots$.

Proof For $n = 1, 2, \ldots$, $P(n,t) = \int_0^\infty p(n,\omega)P(0,t-\omega)\, d\omega$
so that the conclusion follows from Lemmas 3.1, 3.2, and 3.3.

QED

3.2 New Results A key theorem follows from these lemmas.

Theorem 3.1 If $f(t)$ is Pff$_k$ with $f(t) = 0$ for $t < 0$, then $P(n,t)$ is PT$_k$, where $n$ ranges over the values $0, 1, 2, \ldots$.

Proof We shall use an inductive proof. First we demonstrate $P(n,t)$ is PT$_2$. Let $n_1 < n_2$, $t_1 < t_2$ and define
\[
D_2 = \begin{vmatrix}
P(n_1,t_1) & P(n_1,t_2) \\
P(n_2,t_1) & P(n_2,t_2)
\end{vmatrix}
\]
(a) Let $n_1 = 0$. If $t_1 > 0$, the first row consists of 0's so that $D_2 = 0$. If $t_1 \leq 0$, $D_2 = P(n_2,t_2) - P(n_1,t_2) \geq 0$.
(b) Let $n_1 > 0$. Then for $t_1 < 0$, $D_2 = P(n_2,t_2) - P(n_1,t_2) \geq 0$. 

- 15 -
For \( t_1 \geq 0 \), \( D_2 = \int \begin{vmatrix} P(n_1,t_1) & P(n_1,t_1-\theta) \\ P(n_1,t_2) & P(n_1,t_2-\theta) \end{vmatrix} |p(n_2-n_1,\theta)| d\theta \geq 0 \)

(3.1)

by Lemma 3.4.

Now assume we have shown \( P(n,t) \) is \( PT_{m-1} \) for \( m \leq k \). We shall show that this implies \( P(n,t) \) is \( PT_m \). From this it will follow that \( P(n,t) \) is \( PT_k \).

Define \( \Delta_j = t_{j+1} - t_j \), \( j = 1, 2, \ldots, m-1 \). For \( t_1 < t_2 < \ldots < t_m \), \( 0 < n_1 < n_2 < \ldots < n_{m-1} \), form

\[
D_m = \begin{vmatrix} P(n,t_1) & P(n+n_1,t_1) & \cdots & P(n+n_{m-1},t_1) \\
P(n,t_2) & P(n+n_1,t_2) & \cdots & P(n+n_{m-1},t_2) \\
\cdots & \cdots & \cdots & \cdots \\
P(n,t_m) & P(n+n_1,t_m) & \cdots & P(n+n_{m-1},t_m) \\
\end{vmatrix}
\]

Let \( n = 0 \).

(a) \( t_1 > 0 \) implies the first column consists of 0's. Hence \( D_m = 0 \).

(b) \( t_1 \leq 0 < t_2 \) implies all entries in the first column are 0, except the first, which is 1. Thus \( D_m > 0 \) by inductive hypothesis.

(c) \( t_1 < t_2 \leq 0 \) implies the first two rows consist of 1's, so that \( D_m = 0 \).

Hence we let \( n > 0 \). Then
$$D_m = \begin{bmatrix}
\int p(n, \theta) P(0, t_1 - \theta) d\theta & \int p(n, \theta) P(n_1, t_1 - \theta) d\theta & \cdots & \int p(n, \theta) P(n_{m-1}, t_1 - \theta) d\theta \\
\int p(n, \theta + \Delta_1) P(0, t_1 - \theta) d\theta & \int p(n, \theta + \Delta_1) P(n_1, t_1 - \theta) d\theta & \cdots & \int p(n, \theta + \Delta_1) P(n_{m-1}, t_1 - \theta) d\theta \\
\int p(n, \theta + \Delta_{m-1}) P(0, t_1 - \theta) d\theta & \int p(n, \theta + \Delta_{m-1}) P(n_1, t_1 - \theta) d\theta & \cdots & \int p(n, \theta + \Delta_{m-1}) P(n_{m-1}, t_1 - \theta) d\theta 
\end{bmatrix}$$
Thus
\[
D_m = \int \ldots \int_{\theta_1 \leq \ldots \leq \theta_m} D' D'' d\theta_1 \ldots d\theta_m \tag{3.2}
\]

where
\[
D' = \begin{vmatrix}
  p(n, \theta_1) & p(n, \theta_2) & \cdots & p(n, \theta_m) \\
  p(n, \theta_1 + \Delta_1) & p(n, \theta_2 + \Delta_1) & \cdots & p(n, \theta_m + \Delta_1) \\
  p(n, \theta_1 + \Delta_{m-1}) & p(n, \theta_2 + \Delta_{m-1}) & \cdots & p(n, \theta_m + \Delta_{m-1})
\end{vmatrix}
\]

and
\[
D'' = \begin{vmatrix}
  P(0, t_1 - \theta_1) & P(n_1, t_1 - \theta_1) & \cdots & P(n_{m-1}, t_1 - \theta_1) \\
  P(0, t_1 - \theta_2) & P(n_1, t_1 - \theta_2) & \cdots & P(n_{m-1}, t_1 - \theta_2) \\
  P(0, t_1 - \theta_m) & P(n_1, t_1 - \theta_m) & \cdots & P(n_{m-1}, t_1 - \theta_m)
\end{vmatrix}
\]

from (13, p.48, prob. 68). \(\frac{m(m-1)}{2}\)

Then \(\text{sgn } D' = 0\) or \((-1)^{\frac{m(m-1)}{2}}\) since:

(a) \(0 > -\Delta_1 > -\Delta_2 > \ldots > -\Delta_{m-1}\).

(b) Reversing the order of the rows in \(D'\) multiplies \(D'\) by \((-1)^{\frac{m(m-1)}{2}}\).

(c) The resulting determinant \(\geq 0\) by Lemma 3.2.

Consider \(D''\).

(a) If \(t_i \leq \theta_i, \ i = 1, 2, \ldots, m-1,\) then all rows of \(D''\) from the \(i^{th}\) row on consist of all "1"s, so that \(D'' = 0\).

(b) If \(t_i > \theta_m,\) then the first column of \(D''\) consists of 0's, so that \(D'' = 0\).

(c) If \(\theta_{m-1} < t_i \leq \theta_m,\) then
\[ D'' = \begin{bmatrix}
0 & P(n_1, t_1 - \Theta_1) & \ldots & P(n_{m-1}, t_1 - \Theta_1) \\
0 & P(n_1, t_1 - \Theta_2) & \ldots & P(n_{m-1}, t_1 - \Theta_2) \\
\vdots & \vdots & \ddots & \vdots \\
0 & P(n_1, t_1 - \Theta_{m-1}) & \ldots & P(n_{m-1}, t_1 - \Theta_{m-1}) \\
1 & 1 & \ldots & 1
\end{bmatrix} \]

By inductive hypothesis, \( \text{sgn } D'' = 0 \) or \((-1)^{m-1} (1 - \frac{m(m-1)}{2}) = (-1)^{m-1} \frac{m(m-1)}{2} \).

Thus, in general, the integrand of (3.2) is \( \geq 0 \). Hence \( P(n, t) \) is \( \text{PT}_m \).

QED

A similar theorem holds for \( p(n, t) \).

**Theorem 3.2** If \( f(t) \) is \( \text{Pff}_k \) having a continuous first derivative, with \( f(t) = 0 \) for \( t < 0 \), then \( p(n, t) \) is \( \text{PT}_k \) where \( n = 1, 2, \ldots \).

**Proof** First we prove \( p(n, t) \) is \( \text{PT}_2 \). Let \( 0 < n_1 < n_2 \), \( t_1 < t_2 \), and \( d_2 = \begin{vmatrix}
p(n_1, t_1) & p(n_1, t_2) \\
p(n_2, t_1) & p(n_2, t_2)
\end{vmatrix} \).

(a) For \( t_1 < 0 \), the first column of \( d_2 \) consists of \( 0 \)'s, so that \( d_2 = 0 \).

(b) Let \( t_1 \geq 0 \). Write

\[
d_2 = \int \begin{vmatrix}
p(n_1, t_1) & p(n_1, t_1 - \Theta) \\
p(n_1, t_2) & p(n_1, t_1 - \Theta)
\end{vmatrix} p(n_2 - n_1, \Theta) d\Theta \geq 0
\]

by Lemma 3.2.

Assume we have shown \( p(n, t) \) is \( \text{PT}_{m-1} \) for \( m \leq k \). We shall show that this implies \( p(n, t) \) is \( \text{PT}_m \). This in turn will imply...
that $p(n,t)$ is $PT_k$.

Let $0 < n_1, < n_2 < \ldots < n_m, t_1 < t_2 < \ldots < t_m$. Define

$$
d_m = \begin{bmatrix}
p(n, t_1) & p(n+n_1, t_1) & \cdots & p(n+n_{m-1}, t_1) \\
p(n, t_2) & p(n+n_1, t_2) & \cdots & p(n+n_{m-1}, t_2) \\
\vdots & \vdots & & \vdots \\
p(n, t_m) & p(n+n_1, t_m) & \cdots & p(n+n_{m-1}, t_m)
\end{bmatrix}.
$$

$$
\Delta_j = t_{j+1} - t_j, \quad j = 1, 2, \ldots, k-1,
$$

$$
p_r(0, t) = \begin{cases} 
  r & \text{for } 0 \leq t \leq \frac{1}{r} \\
  0 & \text{otherwise}
\end{cases}
$$

and

$$
d_{m,r} = \begin{bmatrix}
\int p(n, \theta)p_r(0, t_1-\theta)d\theta & p(n+n_1, t_1) & \cdots & p(n+n_{m-1}, t_1) \\
\int p(n, \theta+\Delta_1)p_r(0, t_1-\theta)d\theta & p(n+n_1, t_2) & \cdots & p(n+n_{m-1}, t_2) \\
\vdots & \vdots & & \vdots \\
\int p(n, \theta+\Delta_{m-1})p_r(0, t_1-\theta)d\theta & p(n+n_1, t_m) & \cdots & p(n+n_{m-1}, t_m)
\end{bmatrix} \quad (3.3)
$$
Then

\[ d_{m,r} = \cdots \int \ldots \int \frac{p(n, \theta)}{p(n, \theta + \Delta \theta)} \, d\theta \]

Where

\[ \int_1^m \frac{p(n, \theta)}{p(n, \theta + \Delta \theta)} \, d\theta \]

by (31, p. 148, prop. 68).
By Lemma 3.2, \( \text{sgn } d' = 0 \) or \( (-1)^{\frac{m(m-1)}{2}} \).

Consider \( d''_r \).

(a) If \( t_1 < \theta_m \), the last row of \( d''_r \) consists of 0's, so that \( d''_r = 0 \).

(b) If \( t_1 > \theta_m + \frac{1}{r} \), the first column of \( d''_r \) consists of 0's, so that \( d''_r = 0 \).

(c) Let \( \theta_m \leq t_1 \leq \theta_m + \frac{1}{r} \). Let \( \bar{N}_r \) = set of \( \theta = (\theta_1, \ldots, \theta_m) \) such that \( 0 \leq \theta_m - \theta_m - \frac{1}{r} \). Then for \( \theta \) not in \( \bar{N}_r \),

\[
d''_r = \begin{vmatrix}
0 & p(n_1, t_1 - \theta_1) & \cdots & p(n_{m-1}, t_1 - \theta_1) \\
0 & p(n_1, t_1 - \theta_2) & \cdots & p(n_{m-1}, t_1 - \theta_2) \\
\vdots & \vdots & \ddots & \vdots \\
0 & p(n_1, t_1 - \theta_{m-1}) & \cdots & p(n_{m-1}, t_1 - \theta_{m-1}) \\
1 & p(n_1, t_1 - \theta_m) & \cdots & p(n_{m-1}, t_1 - \theta_m)
\end{vmatrix}
\]

By inductive hypothesis, the last determinant is 0 or has sign \( (-1)^{\frac{m(m-1)}{2}} \).

Thus for \( \theta \) not in \( \bar{N}_r \), the integrand \( d' d''_r \geq 0 \). Hence

\[
d_{m,r} \geq \int_{\bar{N}_r} d' d''_r d \theta_1 \ldots d \theta_m,
\]

with the measure of

\[
\bar{N}_r \leq (t_1 - \theta_m - \frac{1}{r})^{m-1}.
\]

We may write, for \( \theta \) in \( \bar{N}_r \),
\[ d \frac{d^n}{d_r} = \begin{vmatrix}
  p(n, \theta_1) & \cdots & p(n, \theta_{m-1}) & \frac{p(n, \theta_m) - p(n, \theta_{m-1})}{1/r} \\
  \vdots & & \vdots & \vdots \\
  p(n, \theta_1 + \Delta_{m-1}) & \cdots & p(n, \theta_{m-1} + \Delta_{m-1}) & \frac{p(n, \theta_m + \Delta_{m-1}) - p(n, \theta_{m-1} + \Delta_{m-1})}{1/r} \\
\end{vmatrix} \\
\]
where \( \delta_r(u) = \begin{cases} 
  1 & \text{for } 0 \leq u \leq \frac{1}{r} \\
  0 & \text{otherwise} 
\end{cases} \)

Since \( 0 \leq \theta_m - \theta_{m-1} \leq \frac{1}{r} \), given \( \epsilon > 0 \), there exists \( r_0 \) such that for all \( r \geq r_0 \),
\[
\left| \frac{p(n, \theta_m) - p(n, \theta_{m-1})}{1/r} \right| \leq \frac{dp(n, \theta)}{d\theta} \bigg|_{\theta = \theta'} + \epsilon, \quad \theta_{m-1} \leq \theta' \leq \theta_m.
\]

Since \( f(t) \) has a continuous first derivative, \( \frac{dp(n, \theta)}{d\theta} \) is continuous on \([0, t_1]\); hence \( \frac{dp(n, \theta)}{d\theta} \bigg|_{\theta = \theta'} + \epsilon \) is bounded by some constant \( M_1 \) for all \( r \geq r_0 \), all \( \theta_m \leq t_1 \). Similarly, the \( i^{th} \) entry of the last column of the first determinant is bounded by some \( M_i \), \( i = 1, 2, \ldots, m \). Since \( p(n, \theta) \) is continuous on \([0, \theta_{m-1} + \Delta_{m-1}]\), a common bound \( M \) exists for all the other entries of the first determinant, for \( 0 \leq \theta_m \leq t_1 \). Hence the first determinant is bounded by \( m! M^{m-1} \max_{i=1, \ldots, m} \), independent of \( r \).

Similarly we can demonstrate a bound independent of \( r \) for the
second determinant. Hence $d' d''$ is bounded, independent of $r$, so that by letting $r \to \infty$, we see that $\int_{N_r} \cdots \int d' d'' d\theta_m \to 0$. Hence

$$\lim_{r \to \infty} d_{m, r} = 0.$$ 

But returning to (3.3), we see that $\lim_{r \to \infty} d_{m, r} = d_{m}$. Hence $p(n, t)$ is $P_{\mathbb{M}}^\ast$.

QED

In addition to their use in the present problem, Theorems 3.1 and 3.2 are of interest in renewal theory. We will discuss further consequences of these theorems in subsequent sections.

As a consequence of Theorem 3.1, we obtain:

**Theorem 3.3** If $f$ is $P_{\mathbb{M}}$ with $f(t) = 0$ for $t < 0$, then $P(n, t)$ is $P_{\mathbb{M}}$ in $n$ for each $t \geq 0$.

**Proof** For $t = 0$, $P(n, t) = \begin{cases} 0 & \text{for } n < 0 \\ 1 & \text{for } n \geq 0 \end{cases}$.

Hence by an argument similar to that of Lemma 3.3, $P(n, t)$ may be proved to be $P_{\mathbb{M}}$.

For $t > 0$, $n_1 < n_2$, and $m_1 < m_2$, let

$$D = \begin{vmatrix} P(n_1-m_2, t) & P(n_1-m_2, t) \\ P(n_2-m_1, t) & P(n_2-m_2, t) \end{vmatrix}.$$ 

(a) If $n_1 \leq m_2$, $P(n_1-m_2, t) = 0$. Thus $D \geq 0$.

(b) Assume $n_1 > m_2$. Thus $m_1 < m_2 < n_1 < n_2$. Then

$$D = \int \begin{vmatrix} P(n_1-m_2, t) & P(n_1-m_2, t) \\ P(n_1-m_1, t-\theta) & P(n_1-m_2, t-\theta) \end{vmatrix} p(n_2-n_1, \theta) d\theta.$$ 

- 24 -
Since the determinant in the integrand $\geq 0$ by Th. 3.1, $D \geq 0$.

\[ QED \]

For $n = 0, 1, 2, \ldots$, define $\Delta P(n,t) = P(n+1,t) - P(n,t)$. We show that:

**Theorem 3.1** If $f$ is Pff$_2$ with $f(t) = 0$ for $t < 0$, then $\Delta P(n,t)$ is Pff$_2$.

**Proof** $\Delta P(n,t) = P(n+1,t) - P(n,t) = \\
\int p(n,\theta) \left\{ P(1,t-\theta) - P(0,t-\theta) \right\} d\theta = \int p(n,\theta) \Delta P(0,t-\theta) d\theta$.

If we show $\Delta P(0,t-\theta)$ is Pff$_2$ in the second argument, then by applying Lemma 3.1, the conclusion will follow.

For $t_1 < t_2$, $\theta_1 < \theta_2$, define

\[
D = \begin{vmatrix}
\Delta P(0,t_1-\theta_1) & \Delta P(0,t_1-\theta_2) \\
\Delta P(0,t_2-\theta_1) & \Delta P(0,t_2-\theta_2)
\end{vmatrix}.
\]

(a) If $t_1 \leq \theta_1$, then $\Delta P(0,t_1-\theta_1) = 1 - 1 = 0$, and $\Delta P(0,t_1-\theta_2) = 1 - 1 = 0$. Thus $D = 0$.

(b) If $\theta_1 < t_1 < t_2 < \theta_2$, then $\Delta P(0,t_1-\theta_2) = 1 - 1 = 0$, and $\Delta P(0,t_2-\theta_2) = 1 - 1 = 0$. Thus $D = 0$.

(c) If $\theta_1 < t_1 < \theta_2 < t_2$, then $\Delta P(0,t_1-\theta_2) = 1 - 1 = 0$, while $\Delta P(0,t_1-\theta_1) \geq 0$ and $\Delta P(0,t_2-\theta_2) \geq 0$. Hence $D \geq 0$.

(d) If $\theta_1 < \theta_2 < t_1 < t_2$, then

\[
D = \begin{vmatrix}
P(1,t_1-\theta_1) & P(1,t_1-\theta_2) \\
P(1,t_2-\theta_1) & P(1,t_2-\theta_2)
\end{vmatrix} \geq 0 \quad \text{by Th. 3.1.}
\]

Note that all possibilities are covered: (a) implies that only $\theta_1 < t_1 < t_2$ need be considered. (b), (c), and (d) respectively show $D \geq 0$ for $t_2 \leq \theta_2$, $t_1 \leq \theta_2 < t_2$, and $\theta_2 < t_1$. 

- 25 -
Thus $\Delta P(n, t)$ is $Pf^2$.

QED

Theorem 3.4 immediately yields:

Theorem 3.5 If $f$ is $Pf^2$ with $f(t) = 0$ for $t < 0$, then

$\Delta P(n, t)$ is $Pf^2$ in $n$.

Proof Let $n_1 < n_2$, $m_1 < m_2$. Define

$D = \begin{vmatrix}
\Delta P(n_1 - m_1, t) & \Delta P(n_1 - m_2, t) \\
\Delta P(n_2 - m_1, t) & \Delta P(n_2 - m_2, t)
\end{vmatrix}.$

(a) If $n_1 < m_2$, $\Delta P(n_1 - m_2, t) = 0$. Thus $D \geq 0$.

(b) Let $n_1 \geq m_2$. Thus $m_1 < m_2 \leq n_1 < n_2$. Then

$D = \int \left| \begin{array}{cc}
\Delta P(n_1 - m_2, t - \theta) & \Delta P(n_1 - m_2, t) \\
\Delta P(n_2 - m_2, t - \theta) & \Delta P(n_2 - m_2, t)
\end{array} \right| p(m_2, \theta) \, d\theta \geq 0$

since the determinant of the integrand $\geq 0$ by Th. 3.4.

QED

Let us now return to the Inventory Problem motivating the whole line of development above.

Theorem 3.6 If the probability densities $f_{ij}(t)$, $j = 1, \ldots, d_i$; $i = 1, \ldots, k$; for the length of life $t$ of a single unit are $Pf^2$, then $q_i(n)$, the probability density for the total number $n$ of units of the $i^{th}$ type failing during $[0, t_0]$, is $Pf^2$ in $n$, for $i = 1, \ldots, k$.

Proof $q_i(n)$ is obtained as the convolution of the densities for the number of replacements of component $i, j$ for $j = 1, \ldots, d_i$.

By Th. 3.4, each such density is $Pf^2$ in $n$, and by Lemma 3.1, the convolution of $Pf^2$ densities is $Pf^2$.

QED
Recall that \( Q_1(n) = \sum_{m=0}^{n} q_1(m) \); i.e., \( Q_1(n) \) represents the probability of experiencing a total of at most \( n \) failures in the \( d_1 \) sockets. Then, we characterize \( Q_1(n) \) in

**Theorem 3.7** If \( f_{ij}(t) \) are Pff\(_2\) for \( j = 1, \ldots, d_1 \), then \( Q_1(n) \) is Pff\(_2\) in \( n \).

**Proof** Write \( Q_1(n) = \sum_{m=0}^{\infty} q_1(m) H'(n-m) \), where

\[
H'(r) = \begin{cases} 
1 & \text{for } r = 0, 1, 2, \ldots \\
0 & \text{for } r = -1, -2, \ldots 
\end{cases}
\]

Then by the same argument as in Lemma 3.3, \( H'(r) \) is Pff\(_2\) in \( r \), where \( r \) takes on integer values. Since \( Q_1(n) \) is the convolution of two Pff\(_2\) functions \( Q_1(n) \) is Pff\(_2\) in \( n \).

QED

As a result, we get

**Theorem 3.8** If the \( f_{ij}(t) \) are Pff\(_2\) for \( j = 1, \ldots, d_1 \), then \( Q_1(n+1)/Q_1(n) \) and \( \Delta R_1(n) \) are decreasing functions of \( n \), where \( n = 0, 1, 2, \ldots \).

**Proof** \( \Delta R_1(n) = \ln Q_1(n+1) - \ln Q_1(n) = \ln \{ Q_1(n+1)/Q_1(n) \} \).

By Theorem 3.7, \( Q_1(n+1)/Q_1(n) \) is decreasing in \( n \). Hence \( \Delta R_1(n) \) is decreasing in \( n \).

QED

The climax of this long line of reasoning may be stated in:

**Theorem 3.9** If the underlying failure densities \( f_{ij} \), \( j = 1, \ldots, d_1 \); \( i = 1, \ldots, k \); are Pff\(_2\), then Procedures 1 and 2 of Section 2.1 yield the optimal spare parts kits. That is, the
hypothesis of Th. 2.1 holds, so that the conclusion and Corollary follow.

The actual computation involved in obtaining the \( Q_1(n) \) in the general case will be tedious, except in the case of exponential life distributions (Section 6.). In general, an electronic computer would probably be needed.

4. EXTENSIONS OF THE THEORETICAL RESULTS.

In this section, we extend some of the theorems of Section 3. Using these extensions, we will see that Procedures 1 and 2 of Section 2.1 yield the solutions to a more general inventory model than the one originally stated.

4.1 Convolution of Non-Identical Densities If we examine the proofs of Th. 3.1 and 3.2, we note that the successive convolutions yielding \( p(n,t) \) and \( P(n,t) \) need not necessarily refer to a common density. Let us therefore define \( p^1(n,t) = f_1 * f_2 * \cdots * f_n(t) \), the n-fold convolution of successive densities \( f_i, i = 1, 2, \ldots \),

\[
\begin{align*}
P^1(n,t) &= \int_t^{\infty} p^1(n,u)du, \text{ and} \\
\Delta P^1(n,t) &= P^1(n+1,t) - P^1(n,t).
\end{align*}
\]

Thus \( p^1(n,t), P^1(n,t), \) and \( \Delta P^1(n,t) \) are analogous to \( p(n,t), P(n,t), \) and \( \Delta P(n,t), \) the only difference being that the successive densities convoluted need not be the same.

We now present generalizations of earlier results, using the newly defined functions.
Lemma 4.1 If \( f_i(t) \) is \( \text{Pff}_k \) with \( f_i(t) = 0 \) for \( t < 0 \),
i = 1, 2, ..., then \( P'(n,t) \) is \( \text{Pff}_k \) in \( t \) for \( n = 1, 2, ... \).

Proof As before, repeated application of Lemma 3.1 yields the conclusion.

Lemma 4.2 If \( f_i(t) \) is \( \text{Pff}_k \) with \( f_i(t) = 0 \) for \( t < 0 \),
i = 1, 2, ..., then \( P'(n,t) \) is \( \text{Pff}_k \) in \( t \) for \( n = 1, 2, ... \).

Proof The proof is as in Lemma 3.4.

Theorem 4.1 If \( f_i(t) \) is \( \text{Pff}_k \) with \( f_i(t) = 0 \) for \( t < 0 \),
i = 1, 2, ..., then \( P'(n,t) \) is \( \text{PT}_k \), where \( n \) ranges over the values 0, 1, 2, ...

Proof The proof consists of the same arguments as in Th. 3.1. The only difference is that \( P' \) and \( P' \) replace \( P \) and \( P \) respectively—with one slight adjustment. In writing the analogue of (3.1), the quantity \( P'(n_2 - n_1, \theta) \) must refer to the convolution of densities \( f_{n_1+1}, ..., f_{n_2} \) (rather than \( f_1, ..., f_{n_2-n_1} \)). Similarly the quantities \( P'(n_1, t_1 - \theta_j) \) appearing in the analogue of (3.2), must refer to the convolutions of densities \( f_{n+1}, ..., f_{n+n_1} \) (rather than \( f_1, ..., f_{n_1} \)). These adjustments do not affect the validity of the arguments however.

Theorem 4.2 If \( f_i(t) \) is \( \text{Pff}_k \) with continuous first derivative, and \( f_i(t) = 0 \) for \( t < 0 \), \( i = 1, 2, ... \), then \( P'(n,t) \) is \( \text{PT}_k \), where \( n \) ranges over the values 0, 1, 2, ...

Theorem 4.3 If \( f_i(t) \) is \( \text{Pff}_2 \) with \( f_i(t) = 0 \) for \( t < 0 \),
i = 1, 2, ..., then \( P'(n,t) \) is \( \text{Pff}_2 \) in \( n \) for each \( t \geq 0 \), where \( P'(n,t) \) is defined = 0 for \( n < 0 \).
Theorem 4.4.1 If $f_i(t)$ is Pff$_2$ with $f_i(t) = 0$ for $t < 0$, $i = 1, 2, \ldots$, then $\Delta P'(n,t)$ is PT$_2$.

Theorem 4.4.5 If $f_i(t)$ is Pff$_2$ with $f_i(t) = 0$ for $t < 0$, $i = 1, 2, \ldots$, then $\Delta P'(n,t)$ is Pff$_2$ in $n$.

The proofs of Theorems 4.2 through 4.5 are as in the corresponding theorems in Section 3, with the adjustments described in the proof of Th. 4.1.

These generalizations permit us to state that Procedures 1 and 2 of Section 2.1 for obtaining the optimal spares kits (See Theorem 3.9.) apply even when the successive replacements of a given component have different (known) failure distributions. At first glance, this may seem like a generalization of little application, since in most cases successive replacements of any one component will be made with units having a common failure distribution. However, if at the beginning of the period, the components already in the system have been aged by previous use, the (conditional) distribution of remaining life is different from the distribution of life of future replacements (except only in the case of exponential life distributions). Another possibility is that replacements may be of different known ages. In this connection, it is helpful to note that the conditional probability density for the remaining life $t$ of a component given its age $t_0$ is Pff$_2$ if the underlying density is Pff$_2$; i.e.:

Theorem 4.6 If $f(t)$ is Pff$_2$ with $f(t) = 0$ for $t < 0$, then
\[ g(t) = \begin{cases} 
0 & \text{for } t < 0 \\
\frac{f(t_0 + t)}{\int_{t_0}^{\infty} f(u) \, du} & \text{for } t \geq 0
\end{cases} \quad \text{is Pff}_2. \]

**Proof** Let \( t_1 < t_2, \; \theta_1 < \theta_2 \). Define

\[ D = \begin{vmatrix}
g(t_1 - \theta_1) & g(t_1 - \theta_2) \\
g(t_2 - \theta_1) & g(t_2 - \theta_2)
\end{vmatrix}, \]

\[ D' = \frac{1}{\int_{t_0}^{\infty} f(u) \, du} \begin{vmatrix}
f(t_0 + t_1 - \theta_1) & f(t_0 + t_1 - \theta_2) \\
f(t_0 + t_2 - \theta_1) & f(t_0 + t_2 - \theta_2)
\end{vmatrix}. \]

Then if \( t_1 \geq \theta_2 \), \( D = D' \). Thus \( \text{sgn } D = \text{sgn } D' \geq 0 \) since \( f \) is Pff\(_2\). If \( t_1 < \theta_2 \), then \( g(t_1 - \theta_2) = 0 \) by definition, so that \( D \geq 0 \).

QED

We close this section with an application of Th. 4.2 to get the following characterization of compound distributions, not related to our Inventory Problem:

**Theorem 4.7** Let \( X_1 \geq 0 \) be distributed with density \( f_1(t) \), a Pff\(_k\), \( i = 1, 2, \ldots \). Define

\[ S_N = \sum_{i=1}^{N} X_i, \quad \text{where } \text{N is a random variable independent of } X_1, \]

\( X_2, \ldots, \) with density \( g(n, \mu) \), where \( \mu \) is a parameter, and \( g \) is PT\(_k\). Then \( r(t, \mu) \), the probability density for \( S_N \), is PT\(_k\).

**Proof** \( r(t, \mu) = \sum_{n=1}^{\infty} p[\text{N=n}] f_1 \star \cdots \star f_n(t) = \sum_{n=1}^{\infty} g(n, \mu) p'(n, t). \)
By Th. 4.1, \( p'(n,t) \) is \( P_{T_k} \). Hence by Lemma 3.1, \( r(t,\mu) \) is \( P_{T_k} \).

QED

For example, consider a Poisson process with parameter \( \lambda \) as describing the occurrence of events. If \( \mu \) is the time elapsed since the start of the process, \( g(n,\mu) = (e^{-\mu} \mu^n) / n! \). If at the time the \( n \)th event occurs, a random variable \( X_n \geq 0 \) with density \( f_n(t) \), a \( P_{f_k} \), is observed, then the cumulative random variable,

\[
S_N = \sum_{i=1}^{N} X_i
\]

has a \( P_{T_k} \) density (in \( t \) and \( \mu \)).

4.2 Scheduled Times Random

Suppose the \( t_{ij} \) are not constants but positive random variables \( T_{ij} \) with corresponding \( P_{f_2} \) densities \( g_{ij}(t), j = 1, \ldots, d_i; i = 1, \ldots, k \). We shall show that when \( d_i = 1, i = 1, \ldots, k \), (only one operating component of each type), the probability density for the number of replacements of component \( i \) during \([0,t_i]\) is \( P_{f_2} \), so that by the arguments of Theorem 3.7, 3.8, and 3.9, we may conclude that Procedures 1 and 2 of Section 2.1 yield optimal spare parts kits, as before.

We shall drop the subscript \( i \) in what follows. Define

\[
\Phi(n,\omega) = \begin{cases} 
0 & \text{for } n < 0 \\
\text{probability that the sum of } n+1 \text{ independent observations from } f(t) \text{ exceeds } T \text{ by at least } \omega, \text{ where } \\
-\infty < \omega < \infty, \text{ for } n = 0, 1, 2, \ldots.
\end{cases}
\]

Then

\[
\Phi(n,\omega) = \int P(n+1,t)g(t-\omega)dt \text{ for } n = 0, 1, 2, \ldots.
\]
Thus, since $P(n,t)$ is $PT_2$ by Th. 3.1 and $g(t)$ is $Pff_2$ by assumption, we conclude that $\Phi(n,\omega)$ is $PT_2$ (where $n$ ranges over $0, 1, 2, \ldots$) by Lemma 3.1.

Next we shall show $\Phi(n,\omega)$ is $Pff_2$ in $n$ for each fixed real $\omega$. Let $n_1 < n_2$, $m_1 < m_2$. Define

$$D = \begin{vmatrix} \Phi(n_1-m_1,\omega) & \Phi(n_1-m_2,\omega) \\ \Phi(n_2-m_1,\omega) & \Phi(n_2-m_2,\omega) \end{vmatrix}$$

(a) If $n_1 < m_2$, then $\Phi(n_1-m_2,\omega) = 0$ by definition. Hence $D \geq 0$.

(b) Assume $m_1 < m_2 < n_1 < n_2$. Thus, we may write

$$D = \int \begin{vmatrix} \Phi(n_1-m_1,\omega-	heta) & \Phi(n_1-m_2,\omega-	heta) \\ \Phi(n_1-m_1,\omega) & \Phi(n_1-m_2,\omega) \end{vmatrix} p(n_2-n_1,\theta) d\theta$$

But the determinant in the integrand $\geq 0$ since $\Phi(n,\omega)$ is $PT_2$ (where $n$ ranges over $0, 1, 2, \ldots$). Hence $\Phi(n,\omega)$ is $Pff_2$ in $n$ for each fixed real $\omega$.

Thus $\Phi(n+1,\omega)/\Phi(n,\omega)$ decreases in $n$ for fixed $\omega$. But $\Phi(n,0)$ represents the probability that $n$ or less spares will be required during $[0,T]$. Using the arguments of Theorems 3.7, 3.8, and 3.9, the monotonicity of the ratio $\Phi(n+1,0)/\Phi(n,0)$ is sufficient to insure that Procedures 1 and 2 yield optimal spare parts kits.

5. EXPONENTIAL FAILURE DISTRIBUTIONS

5.1 Explicit Solution In the case of a complex electronic system, the component failure distribution usually assumed is the
exponential. We shall now obtain an explicit solution of the Inventory Problem (Section 1.1) for this case, and present an illustration of it.

Assume then, that \( f_{ij}(t) = \begin{cases} 0 & \text{for } t < 0 \\ \mu_{ij} e^{-\mu_{ij} t} & \text{for } t \geq 0 \end{cases} \),

where \( \mu_{ij} \) represents the failure rate per unit of time, \( j = 1, \ldots, d_i; i = 1, \ldots, k \). It follows immediately that

\[
\Delta p_{ij}(n, t_{ij}) = e^{-\mu_{ij} t_{ij} (\mu_{ij} t_{ij})^n / n!} \text{, a Poisson density with parameter } \mu_{ij} t_{ij} \text{ (1, p.272). Next note that}
\]

\[
q_i(n) = e^{-\lambda_i n / n!} \quad (5.1)
\]

where \( \lambda_i = \sum_{j=1}^{d_i} \mu_{ij} t_{ij} \), since the convolution of Poisson densities is a Poisson density with parameter given by the sum of the separate parameters (2, p.205).

We know that the exponential density is \( P_{f\infty} (9, p.125) \). Hence Procedures 1 and 2 yield the optimal spare parts kits by Th. 3.9. In this connection, it is interesting to give a separate proof that \( \Delta R_i(n) \) is a decreasing function of \( n \) for fixed \( t \), the condition needed to apply Th. 3.9.

**Theorem 5.1** If \( f_{ij}(t) = \begin{cases} 0 & \text{for } t < 0 \\ \mu_{ij} e^{-\mu_{ij} t} & \text{for } t \geq 0 \end{cases} \),

\( j = 1, \ldots, d_i \), then \( \Delta R_i(n) \) is a decreasing function of \( n \).  

**Proof** Dropping subscripts,
\[ \Delta R(n) = \ln \left\{ \frac{Q(n+1)}{Q(n)} \right\} = \ln \left\{ 1 + \frac{\lambda^{n+1}/(n+1)!}{\sum_{j=0}^{\infty} \left[ \lambda^j/j! \right]} \right\} \]

It will be sufficient to show that

\[ g(n, \lambda) = \frac{\lambda^{n+1}/(n+1)!}{\sum_{j=0}^{n} \left[ \lambda^j/j! \right]} \]

is a decreasing function of \( n \) for all \( \lambda > 0 \).

Now \( g(n, \lambda) - g(n-1, \lambda) \) has the same sign as

\[ f(n, \lambda) = \lambda \sum_{j=0}^{n-1} \lambda^j/j! - (n+1) \sum_{j=0}^{n} \lambda^j/j! . \]

But, after simplification,

\[ \frac{df(n, \lambda)}{d\lambda} = f(n-1, \lambda) \quad (5.2) \]

with \( f(1, \lambda) = -2 - \lambda < 0 \) for \( \lambda > 0 \). Suppose \( f(n, \lambda) < 0 \) for \( n = 1, 2, \ldots, n_0 - 1 \). Then \( \frac{df(n_0, \lambda)}{d\lambda} < 0 \) for \( \lambda > 0 \) by (5.2).

Since \( f(n_0, 0) = -(n_0 + 1) < 0 \) and \( \frac{df(n_0, \lambda)}{d\lambda} < 0 \), then \( f(n_0, \lambda) < 0 \) for all \( \lambda > 0 \). By induction, \( f(n, \lambda) < 0 \) for \( n = 1, 2, \ldots; \lambda > 0 \). Thus \( g(n, \lambda) \) is a decreasing function of \( n \) for \( \lambda > 0 \).

A useful approximation in the case of exponential failure distributions may be derived as follows.

\[ \Delta R_1(n) = \ln \left\{ 1 + \frac{\lambda_i^{n+1} e^{-\lambda_i}}{(n+1)!} \sum_{j=0}^{n} \frac{\lambda_i^j e^{-\lambda_i}}{j!} \right\} . \]

For \( n \) such that \( \sum_{j=0}^{n} \frac{\lambda_i^j e^{-\lambda_i}}{j!} \) is close to 1, and hence
\[
\frac{\lambda_i^{n+1} e^{-\lambda_i}}{(n+1)!} \quad \text{close to 0, we may write}
\]
\[
\Delta R_i(n) \approx \frac{\lambda_i^{n+1} e^{-\lambda_i}}{(n+1)!} / \sum_{j=0}^{n} \frac{\lambda_i^j e^{-\lambda_i}}{j!}, \text{ so that}
\]
\[
\Delta R_i(n) \approx \frac{\lambda_i^{n+1} e^{-\lambda_i}}{(n+1)!} \quad (5.3)
\]

Since the latter expression is tabulated (11), the computation of Procedures 1 or 2 is simple even with only a desk calculator.

5.2 Illustration A UHF receiving system and a VHF receiving system are to be placed in the field for a three months period of experimentation. The essential tubes in the two systems are described in Table 1.

During the period of operation in the field, the UHF tubes are each scheduled for 332 hours of use and the VHF tubes are each scheduled for 2160 hours of use. Assuming an exponential life distribution for each of the tube types with failure rate as shown above, and assuming independence of operation of the tubes, find an optimal allocation of spare parts for various spares budgets, i.e., an allocation which maximizes assurance of continued system operation during the period in the field.

First, we compute the expected number of spares of each type used during the period:

\[
\lambda_1 = \frac{1}{2500} \left\{ 4.332 + 4.2160 \right\} = 4.0
\]
\[
\lambda_2 = \frac{1}{4000} \left\{ 2.332 + 5.2160 \right\} = 2.9
\]
### TABLE 1

<table>
<thead>
<tr>
<th></th>
<th>Tube Type</th>
<th>Failure rate/hour</th>
<th>$c_i, \text{ Cost per tube ($)}</th>
<th>Number in UHF, scheduled for 332 hours of use each</th>
<th>Number in VHF, scheduled for 2160 hours of use each</th>
<th>$\lambda_i, \text{ Expected number of failures}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Radechon</td>
<td>$\frac{1}{2500}$</td>
<td>$240$</td>
<td>$4$</td>
<td>$4$</td>
<td>$4.0$</td>
</tr>
<tr>
<td>2</td>
<td>Memotron</td>
<td>$\frac{1}{4000}$</td>
<td>$1025$</td>
<td>$2$</td>
<td>$5$</td>
<td>$2.9$</td>
</tr>
<tr>
<td>3</td>
<td>Carcinotron</td>
<td>$\frac{1}{800}$</td>
<td>$1158$</td>
<td>$4$</td>
<td>$0$</td>
<td>$1.7$</td>
</tr>
<tr>
<td>4</td>
<td>TWT</td>
<td>$\frac{1}{6000}$</td>
<td>$750$</td>
<td>$2$</td>
<td>$0$</td>
<td>$0.11$</td>
</tr>
</tbody>
</table>

\[ \lambda_3 = \frac{1}{800} \quad \left\{4 \cdot 332\right\} = 1.7 \]

\[ \lambda_4 = \frac{1}{6000} \quad \left\{2 \cdot 332\right\} = 0.11 \]

One way to determine the first value of $r$ to use is to compute $\lambda_1 + 3\sqrt{\lambda_1}$ and round to the nearest integer, obtaining 10. Let $n_1^* = 10$. Thus $n_1^*$ corresponds to a value three standard deviations above the mean. (In a Poisson distribution, the standard deviation equals the square root of the mean.)

Using the approximation (5.3), we let $r$ be determined from

\[ r = \frac{1}{c_1} e^{-\lambda_1} \frac{\lambda_1^{10}}{10!} = 0.000022. \text{ (Molina's Table I, 11.)} \]

We then find $n_2^*$ as the largest value of $n$ such that

\[ \frac{1}{c_2} e^{-\lambda_2} \frac{\lambda_2^n}{n!} \geq 0.000022 \quad (5.4) \]
Using Molina's Table I (11), we find \( n_2^* = 6 \).

Replacing the subscript 2 in (5.4) by 3 and 4 respectively and proceeding similarly, we find \( n_3^* = 4 \) and \( n_4^* = 1 \).

From \( \overline{n}^* = 10, 6, 4, 1 \), we compute

\[
Q(n^*) = \prod_{i=1}^{4} \sum_{x=0}^{n_i^*} e^{-\lambda_i} \frac{\lambda_i^x}{x!} = .935, \text{ and}
\]

\[
c(n^*) = \sum_{i=1}^{4} c_i n_i^* = \$13,932.
\]

Thus to obtain maximum assurance of continued system operation under a budget of \( \$13,932 \) for spares of the four tube types, we would stock 10 radechons, 6 memotrons, 4 Carcinotrons, and 1 TWT. The assurance obtained would be .935.

By taking \( n_1^* = 8, 9, 11, 12, \) and 13 respectively, and proceeding in a similar fashion, we would obtain the other five points plotted in Figure 1. Thus Figure 1 shows the maximum assurance of continued system operation obtainable for various given budgets for spares. In addition, the composition of the spares kit yielding the plotted maximum assurance is shown next to each point. Note that additional points lying between those shown on the optimal curve of Figure 1 may be computed as needed.

Given any value of \( c_0 \), we may read off Figure 1 the protection and composition of the optimal spare parts kit of cost not exceeding \( c_0 \). It would be necessary to compute two successive optimal points, \( \overline{n}^* \) and \( n^*(\alpha) \), on the curve such that \( c(n^*) \leq c_0 \).
while $c(n^*) > c_0$. $n^*$ would then constitute the solution, the error being at most $Q(n^*) - Q(n)$.

5.3 Application To Reliability Design  It is interesting to note that the solution of the Inventory Model of Section 1.1 may be applied to a similar model in the field of reliability. Suppose we are designing a complex system (say a missile); we wish to attain maximum reliability by providing redundant units within a weight restraint. The statement of the Inventory Problem then describes the present reliability problem if we make certain minor modifications. Simply substitute "redundant standby unit" for "spare", "weight" for "cost". We assume, too, that replacement is made with perfect reliability.

It is then clear that the Inventory Problem and the present reliability problem are mathematically the same. Hence Procedures 1 and 2 of Section 2.1 yield the required number of redundant units of each component type to provide maximum reliability under a weight restraint.

6. CONVOLUTIONS WHEN THE RANDOM VARIABLE IS NOT NECESSARILY POSITIVE

The question naturally arises: Do the theorems of Sections 3 and 4 hold when $f(t)$ is not necessarily 0 for $t < 0$? If not, what theorems do hold? In this section, we give a theorem somewhat analogous to Theorem 3.2.

It will be seen that when the density is not 0 for negative argument, successive convolutions have considerably weaker properties.
First we prove

**Theorem 6.1** Let $f(t)$ be $Pf_m$, with $f(t)$ not necessarily 0 for $t < 0$. Let $h_k(t) = \sum_{i=1}^{k} a_i p(n_i, t)$, where $n_1 < n_2 < \ldots < n_k$, $k \leq \frac{m+1}{2}$, and the $a_i$ are real valued constants. Then $h_k(t)$ has $\leq 2(k-1)$ sign changes.

**Proof** Let $V(g)$ designate the number of sign changes of a function $g(t)$ as $t$ ranges over the real line.

(a) For $k = 1$, $V \left\{ a_1 p(n_1, t) \right\} = 0$, so that the theorem holds.

(b) Assume the theorem holds for $k = k_0 - 1$, where $k_0 \leq \frac{m+1}{2}$.

We shall show that this implies that the theorem holds for $k = k_0$.

Write

$$h_{k_0}(t) = \sum_{i=1}^{k_0} a_i p(n_i, t) = \sum_{i=2}^{k_0} a_i \int p(n_i - n_1, \theta) p(n_1, \theta) \, d\theta +$$

$$+ a_1 \lim_{r \to \infty} \int p_r(0, \theta) p(n_1, t - \theta) \, d\theta,$$

where

$$p_r(0, t) = \begin{cases} r & \text{for } 0 \leq t \leq \frac{1}{r} \\ 0 & \text{otherwise} \end{cases}$$

Factoring, we get

$$h_{k_0}(t) = \lim_{r \to \infty} \int \left\{ \sum_{i=2}^{k_0} a_i p(n_i - n_1, \theta) + a_1 p_r(0, \theta) \right\} p(n_1, t - \theta) \, d\theta$$

By inductive hypothesis $\sum_{i=2}^{k_0} a_i p(n_i - n_1, \theta)$ has at most $2(k-2)$
sign changes as a function of $\theta$. By taking $r$ sufficiently large \( a_1 p_r(0, \theta) \) can introduce at most 2 additional sign changes. Thus
\[
\sum_{i=2}^{k} a_i p(n_i, t, \theta) + a_1 p_r(0, \theta)
\]
for sufficiently large $r$, has at most $2(k_0 - 1)$ sign changes. Since $p(n_1, t, \theta)$ is $\text{Pff}_m$, the integral on the right of (6.1) has at most $2(k_0 - 1)$ sign changes as a function of $t$. Taking the limit as $r \to \infty$, the number of sign changes does not increase. Hence $V(h_{k_0}) \leq 2(k_0 - 1)$.

By induction, the theorem holds for $k = 1, 2, \ldots, \frac{m+1}{2}$.

QED

Theorem 6.1 furnishes an interesting analogue to the fundamental variation diminishing property of a Polya type function. From Theorem 3.2 we know that $p(n, t)$ is $\text{Pf}_m$ when $f(t)$ is $\text{Pff}_m$ with $f(t) = 0$ for $t < 0$, with continuous first derivative.

Therefore $h_k(t) = \sum_{i=1}^{k} a_i p(n_i, t)$ has at most $k - 1$ sign changes.

Theorem 6.1 thus tells us that when the restriction $f(t) = 0$ for $t < 0$ is dropped, the upper bound on the number of sign changes of $h_k(t)$ becomes $2(k - 1)$ instead of $k - 1$.

Next let us use Theorem 6.1 to characterize the sign changes of $D(t_1, \ldots, t_k) = |p(n_i, t_j)|$ as a function of $t_1$ say, for fixed $t_2, \ldots, t_k, n_1, \ldots, n_k$, in the general case $f(t)$ not necessarily 0 for $t < 0$. Recall that when $f(t) = 0$ for $t < 0$, $D(t_1, \ldots, t_k)$ changes sign at most $k - 1$ times for fixed $t_2, \ldots, t_k, n_1, \ldots, n_k$.
as $t_1$ varies over the whole real line (Theorem 3.2). Dropping
the restriction $f(t) = 0$ for $t < 0$, we get the somewhat weaker
characterization of Theorem 6.2:

**Theorem 6.2** Let $f(t)$ be $Pf^m$ with $f(t)$ not necessarily 0
for $t < 0$. For fixed $t_2, \ldots, t_k, n_1, \ldots, n_k$, $k \leq \frac{m+1}{2}$;

$$D(t_1, \ldots, t_k)$$

changes sign at most $2(k-1)$ times as $t_1$ ranges
over $(-\infty, +\infty)$.

**Proof** Write

$$D = \sum_{i=1}^{k} p(n_i, t_1) \cdot \text{cof } p(n_i, t_1)$$

where $\text{cof } p(n_i, t_1)$ is the cofactor of $p(n_i, t_1)$ in $D$, $i = 1, \ldots, k$.

But $\text{cof } p(n_i, t_1)$ may be considered a constant since it is a func-
tion of $t_2, \ldots, t_k$ and not of $t_1$; hence we may apply Th. 6.1 to
conclude that $D$ has at most $2(k-1)$ sign changes as $t_1$ ranges over
the whole real line.

**QED**

Similar characterizations may be stated for $D(t_1, \ldots, t_k)$
when $t_j$ varies while the remaining $t$'s are fixed.

Finally, it is apparent that Theorems 6.1 and 6.2 hold when
$p(n, t)$ is replaced by $p'(n, t)$, the convolution of different $Pf^m$
densities, $f_1, f_2, \ldots, f_n$. 

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