CONTINUOUS TIME STOCHASTIC STORAGE PROCESSES

WITH RANDOM LINEAR INPUTS AND OUTPUTS

by

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Rupert G. Miller, Jr.

1. Introduction

The probabilistic treatment of various types of storage systems has received increased attention in the last decade through the impetus of problems in the theory of queues, dams, and inventory systems. Some of the more recent work has been summarized by J. Gani [5], but otherwise the material is scattered over a wide range of journals and books. Outstanding contributions to the theory of storage systems have been made by Arrow, Harris, and Marschak [1], Arrow, Karlin, and Scarf [2], D. G. Kendall [11], [13], P. A. P. Moran [14], [15], and L. Takács [17] to mention just a few.

The behavior of a storage process is determined by the input process and the output mechanism. Various models have been proposed to analyze a system in discrete or continuous time by imposing varied assumptions on the input and output, as for example gamma-type input and constant linear release (by Moran [15] and Kendall [13]). Amidst the vast amounts of recent literature one rather simple model has been overlooked, and the purpose of this paper is to analyze it. Informally the model is described as follows: Let $X(t)$ be the level of the storage process at time $t$; $X(0) = 0$. Starting at time 0 the level
X(t) increases linearly for a random length of time after which it decreases linearly for a random time. Subsequently, the process continues to alternate between periods of increase and decrease. All time intervals are independently distributed, and the periods of increase and decrease are identically distributed, respectively. Should X(t) return to 0 at some time t > 0 it remains at this level until the end of the current period of decrease.

The fluctuations between increase and decrease can be interpreted in three ways. The first interpretation is that periods of input and output actually alternate and never overlap. Another is that there is always a constant linear output to the system provided the level is greater than zero but the input process shuts on and off. When there is input to the system, it is linear and exceeds the output so that the increase is actually the net increase to the system. The third interpretation reverses the constancy of input and output, i.e., constant input but alternating output.

This model was first introduced by Gaver and Miller [6], and the properties of X(t) were investigated for a special case (cf. Section 2). However, the model is actually a derivative of a diffusion model considered earlier by G. I. Taylor [19], S. Goldstein [7], and H. C. Gupta [8]. In their model the barrier at 0 is missing so that X(t) can assume negative values. This was designed to describe certain types of physical diffusion processes rather than storage systems. The inclusion of the barrier at 0 changes the structure of
the process to that of a unilateral diffusion process. Recently, Gupta [9] has treated certain aspects of a related unilateral process.

The storage process under consideration is defined by two sequences of random variables. Let $\xi, \xi_1, \xi_2, \xi_3, \ldots$ be independent positive random variables identically distributed according to $F(\cdot)$, and let $\eta, \eta_1, \eta_2, \eta_3, \ldots$ be independent positive random variables identically distributed according to $G(\cdot)$. The variables $\xi_i, \eta_j$ will be assumed independent. The $\xi_i$ will define the random intervals of increase and the $\eta_j$ the random intervals of decrease. For simplicity $F$ and $G$ will be assumed absolutely continuous with continuous densities $f(\cdot)$ and $g(\cdot)$, respectively, although this will not be crucial to most of the arguments. $P(X(0) = 0) = 1$. If the initial period is an interval of increase, then for $t \geq 0, \Delta t > 0$

\[
X(t + \Delta t) = \begin{cases} 
X(t) + \Delta t & \text{for } \sum_{i=1}^{n}(\xi_i + \eta_i) \leq t < t + \Delta t \leq \sum_{i=1}^{n}(\xi_i + \eta_i) + \xi_{n+1}, \\
\max(0, X(t) - \Delta t) & \text{for } \sum_{i=1}^{n}(\xi_i + \eta_i) + \xi_{n+1} \leq t < t + \Delta t \leq \sum_{i=1}^{n+1}(\xi_i + \eta_i) \\
& n = 0, 1, 2, \ldots,
\end{cases}
\]

with probability 1, and if the initial period is an interval of decrease
(i.e., $X(t)$ sojourns at 0 initially), then for $t \geq 0$, $\Delta t > 0$,

$$X(t + \Delta t) = \begin{cases} \max(0, X(t) - \Delta t) & \text{for } \sum_{i=1}^{n}(\eta_{i} + \xi_{i}) \leq t + \Delta t < \sum_{i=1}^{n}(\eta_{i} + \xi_{i}) + \eta_{n+1}, \\
X(t) + \Delta t & \text{for } \sum_{i=1}^{n}(\eta_{i} + \xi_{i}) + \eta_{n+1} \leq t + \Delta t \leq \sum_{i=1}^{n+1}(\eta_{i} + \xi_{i}), 
\end{cases} \quad n = 0, 1, 2, \ldots,$$

with probability 1. In any calculation the initial interval will be specified.

For ease of exposition the rates of increase and decrease have been assumed to be $\pm 1$. This is arbitrary and not essential to the arguments. The entire discussion can be straightforwardly extended to the case of arbitrary and not necessarily absolutely equal slopes.

The symbol $F/G/\infty$ will be assigned to the above process. (The notation is reminiscent of that used in queueing and mutation processes.) The first and second entries refer to general increase and decrease distributions. Should one or both of them be Markovian (i.e., negative exponential) its or their symbols will be replaced by the letter $M$. The symbol $\infty$ indicates there is no upper barrier on the height of the process. If $\infty$ is replaced by the letter $K$, this will mean that there is an upper reflecting barrier at height $K$ similar to the lower reflecting barrier at 0.
In all cases which will be considered in this paper either 
F or G or both will be Markovian. Of course, the process can be 
studied in greater detail when both are Markovian. Discussion of the 
case of general F and G will be omitted since very few quantitative 
results could be obtained.
2. F/M/∞.

Let \( G(t) = 1 - e^{-\lambda t}, t > 0, \lambda > 0, \) and let the first random interval be \( \eta_1, \) i.e., initially \( X(t) \) remains at zero.

For any value of \( t \) define the random variable \( Y(t) \) by

\[
Y(t) = \begin{cases} 
  t - \sum_{i=1}^{n} (\eta_{i+1}) - \eta_{n+1} & \text{for } \sum_{i=1}^{n} (\eta_{i+1}) + \eta_{n+1} \leq t < \sum_{i=1}^{n+1} (\eta_{i+1}) + \eta_{n+2}, \\
  0 & \text{for } t < \eta_1.
\end{cases}
\]

The value of \( Y(t) \) is the amount of time that has elapsed since the beginning of the last interval of increase before time \( t. \) Because \( F \) is general it is necessary to keep account of \( Y(t) \) in order to have a Markov process. Although the value of \( X(t) \) is the primary concern, the state of the process will actually be defined by three variables: the value of \( X(t) \), the value of \( Y(t) \), and whether \( X(t) \) is increasing or decreasing at \( t. \) This definition of the state variable for the stochastic process makes it a Markov process.

Let \( F_-(x, t) = P(X(t) \leq x, X(t) \text{ decreasing}) \) and

\[
F_+(x, y, t) dy = P(X(t) \leq x, y \leq Y(t) \leq y + dy, X(t) \text{ increasing}).
\]

Since initially \( X(t) \) remains at zero, the assumption of a density with respect to \( y \) in the latter definition is justified. Let \( \varphi_-(s, t) \) be the Laplace-Stieltjes transform of \( F_-(x, t) \) with respect to \( x \) and \( \varphi_-(s, u) \) be the Laplace transform of \( \varphi_-(s, t) \) with respect to \( t. \)
Let $\phi_+(s, y, t)$ denote the Laplace-Stieltjes transform of $F_+(x, y, t)$ with respect to $x$ and $\phi_+(s, v, u)$ the Laplace transform of $\phi_+(s, y, t)$ with respect to $y$ and $t$.

The transforms $\phi_-$ and $\phi_+$ can be derived through consideration of the change in the process in an infinitesimal interval of time. For $\Delta t > 0$,

\begin{equation}
(2.1) \ F_-(x, t + \Delta t) = (1 - \mu \Delta t)F_-(x + \Delta t, t) \\
\quad + \Delta t \int_0^\infty F_+(x + \Delta t, y, t)f^*(y)dy + o(\Delta t),
\end{equation}

where $f^*(y) = f(y)/(1 - F(y))$, and for $x > 0, y > 0$,

\begin{equation}
(2.2) \ F_+(x, y, t + \Delta t) = (1 - f^*(y - \Delta t)\Delta t)F_+(x - \Delta t, y - \Delta t, t) + o(\Delta t).
\end{equation}

The function $F_-$ has a jump discontinuity along the $x$-axis, but assume it to be continuously differentiable with respect to $x$ and $t$ for $x, t > 0$. Also assume $F_+$ to be continuously differentiable with respect to $x, y, t$ for $0 < y < x, 0 < y < t$. Passage to the limit ($\Delta t \to 0$) in (2.1) and (2.2) produces the forward Kolmogorov equations:

\begin{equation}
(2.3) \ \frac{\partial F_-(x, t)}{\partial t} = \frac{\partial F_-(x, t)}{\partial x} - \mu F_-(x, t) + \int_0^\infty F_+(x, y, t)f^*(y)dy,
\end{equation}

for $x, t > 0$, and
(2.4) \[ \frac{\partial F_+(x, y, t)}{\partial t} = -\frac{\partial F_+(x, y, t)}{\partial y} - \frac{\partial F_+(x, y, t)}{\partial x} - f(y)F_+(x, y, t), \]

for \( 0 < y < x, 0 < y < t \). The boundary conditions are

\[ F_+(x, 0, t) = \mu F_-(x, t), \]

(2.5) \[ F_-(x, 0) = 1 \text{ for all } x, \]

(2.6) \[ F_+(x, y, 0) = 0 \text{ for all } x, y. \]

The solution \( F_+ \) to the linear, first order partial differential equation (2.4) subject to the boundary condition (2.5) is given by

\[ F_+(x, y, t) = \mu(1 - F(y))F_-(x - y, t - y). \]

(2.8)

Substitution of (2.8) into (2.3) yields the integro-differential equation

\[ \frac{\partial F_-(x, t)}{\partial t} = \frac{\partial F_-(x, t)}{\partial x} - \mu F_-(x, t) + \mu \int_0^\infty F_-(x - y, t - y)f(y)dy. \]

(2.9)

This equation is similar to the Takács equation [17] for the waiting time in a queue; the only difference lies in the integrand. The Laplace transform \( \Phi_- \) of the solution to (2.9) satisfying in addition (2.6) is
obtained by taking the transform of (2.9) with respect to $x$ and $t$:

$$(2.10) \quad \Phi_-(s, u) = \frac{1 - s\tilde{F}_-(0, u)}{u - s + u(1 - \tilde{F}(u + s))},$$

where $\tilde{F}$ is the Laplace-Stieltjes transform of $F$ and

$$(2.11) \quad \tilde{F}_-(0, u) = \int_0^\infty e^{-ut}\tilde{F}_-(0, t)dt.$$  

From (2.8), $\Phi_+(s, v, u)$ is related to (2.10) by

$$(2.12) \quad \Phi_+(s, v, u) = u\Phi_-(s, u)\left[\frac{1 - \tilde{F}(s + u + v)}{s + u + v}\right].$$

For the complete determination of the transforms (2.10) and (2.12) the function $\tilde{F}_-(0, u)$ must be specified. This can be accomplished through consideration of the distribution of non-empty periods which are analogous to busy periods in the theory of queues. A non-empty period commences each time the storage level leaves zero and becomes positive and ends at the first subsequent return of the level to zero. Its distribution is readily found from the distribution of a corresponding busy period. Let $T$ be the length of a particular realization of a non-empty period, and let $T^*$ be the length of time this sample realization spends decreasing. Then $T$ and $T^*$ are related by $T = 2T^*$. But the distribution of $T^*$ arises in the theory of queues since it corresponds to the length of a non-empty period for a process with jump.
increases and linear output. (For further detail see [17].) Let \( \tilde{T}(s) \) and \( \tilde{T}^*(s) \) denote the Laplace-Stieltjes transforms of the distribution of \( T \) and \( T^* \), respectively. Takács [17] has characterized \( \tilde{T}^* \) as the unique solution of a functional equation. Since \( \tilde{T} \) is simply related to \( \tilde{T}^* \) by \( \tilde{T}(s) = \tilde{T}^*(2s) \), the characterization of \( \tilde{T}^* \) shows that \( \tilde{T}(s) \) is the unique analytic solution to the functional equation

\[
(2.13) \quad f(s) = \tilde{F}(2s + \mu(1 - f(s))),
\]

subject to either of the conditions \( \lim_{s \to \infty} f(s) = 0 \) or \( |f(s)| < 1 \) for \( s > 0 \). The relation (2.13) is very useful in computing the moments of the random variable \( T \) and in certain simple cases will yield an explicit expression for the distribution of \( T \).

The transform \( \tilde{F}(s) \) is closely allied with the required transform \( \tilde{F}_-(0, u) \) as well as being of interest \underline{per se}. The connection between the two transforms has been derived by Takács [17] and is given by

\[
(2.14) \quad \tilde{F}_-(0, u) = \frac{1}{u + \mu(1 - \tilde{T}(u))}.
\]

Hence, if (2.13) can be solved explicitly for \( \tilde{T}(u) \), the expression (2.10) is explicit.
If \( \lambda/\mu > 1 \) where \( \lambda = 1/E(t) \), the drift of the process is towards the zero barrier so \( X(t) \) will have a limiting distribution as \( t \to \infty \), i.e., \( \lim_{t \to \infty} F_-(x, t) \) and \( \lim_{t \to \infty} F_+(x, y, t) \) exist and

\[
F_-(\infty, \infty) + \int_0^\infty F_+(\alpha, y, \infty) \, dy = 1.
\]

This can be substantiated by general renewal arguments (see W. L. Smith [16]). The limits of the transforms can be calculated from (2.10), (2.12), and (2.14) with the aid of Karamata's tauberian theorem (see D. V. Widder [20]):

\[
\varphi_-(s, \infty) = \lim_{u \to 0} \frac{u \varphi_-(s, u)}{1 + u/\lambda} \cdot \frac{1}{1 - \frac{u}{s} (1 - \hat{F}(s))},
\]

(2.15)

\[
\int_0^\infty e^{-vy} \varphi_+(s, y, \infty) \, dy = \lim_{u \to 0} \frac{u \varphi_+(s, v, u)}{\mu \varphi_-(s, \infty)} \left( \frac{1 - F(s + v)}{s + v} \right).
\]

(2.16)

The relation (2.16) is equivalent to

\[
F_+(x, y, \infty) = \mu(1 - F(y)) F_-(x - y, \infty)
\]

(2.17)

so there is actually only one inversion required, namely the inversion of \([1 - (\mu/s)(1 - \hat{F}(s))]^{-1}\). This inversion can be performed, and \(F_-(x, \infty)\) can be written as

\[
F_-(x, \infty) = \frac{1 - \mu/\lambda}{1 + \mu/\lambda} \sum_{n=0}^{\infty} \left( \frac{\lambda}{\mu} \right)^n H(n)(x),
\]

(2.18)
where

\[ H(x) = \lambda \int_0^x (1 - F(y))dy \]  

and \( H^{(n)} \) denotes the n-fold convolution of \( H \). Unfortunately, only for the simplest cases is the series in (2.16) summable to a nice closed form. These limits as \( t \to \infty \) were obtained earlier in [6].

The limiting distribution (provided \( \lambda/\mu > 1 \)) for certain related stochastic storage processes can be obtained directly from (2.15) and (2.16). For example, the storage process with jump inputs (i.e., shot effects) and linear output has a limiting distribution whose transform is given by (2.15) except for renormalization so that \( \varphi(-0, \infty) = 1 \). This distribution is the well-known Pollaczek-Khintchine distribution from the theory of queues. Another example is the storage process with linear input but jump outputs or discharges. Any excess of discharge over storage level is erased, i.e., the level is never negative. The transform of the limiting distribution for this process is (2.16) renormalized.
3. \( M/G/\infty \).

Let \( F(t) = 1 - e^{-\lambda t} \), \( t > 0, \lambda > 0 \), and let the first random interval be \( \xi_1 \), i.e., \( X(t) \) is increasing initially.

For any value of \( t \) now define the random variable \( Y(t) \) to be

\[
Y(t) = \begin{cases} 
  t - \sum_{l=1}^{n} (\xi_l + \eta_l) - \xi_{n+1} & \text{for } \sum_{l=1}^{n} (\xi_l + \eta_l) + \xi_{n+1} \leq t < \sum_{l=1}^{n+1} (\xi_l + \eta_l) + \xi_{n+2}, \\
  0 & \text{for } t < \xi_1.
\end{cases}
\]

The value of \( Y(t) \) is the amount of time that has elapsed since the beginning of the last interval of decrease before time \( t \). Its function is analogous to that of the \( Y(t) \) in the last section; i.e., it serves to make the process Markovian.

Let \( F_+(x, t) = P[X(t) \leq x, X(t) \text{ increasing}] \) and \( \tilde{F}_+(x, u) \) be the Laplace transform of \( F_+(x, t) \) with respect to \( t \). Let \( F_-(x, y, t) \) be \( P[X(t) \leq x, y \leq Y(t) \leq y + dy, X(t) \text{ decreasing}] \) and \( \tilde{F}_-(x, y, u) \) be the Laplace transform of \( F_-(x, y, t) \) with respect to \( t \).

The functions \( \tilde{F}_+ \) and \( \tilde{F}_- \) can be derived from the forward Kolmogorov equations. For \( \Delta t > 0, x > 0 \)

\[
(3.1) \quad F_+(x, t + \Delta t) = (1 - \lambda \Delta t) F_+(x - \Delta t, t) + \Delta t \int_{0}^{\infty} F_-(x - \Delta t, y, t) g^*(y) dy + o(\Delta t),
\]

where \( g^*(y) = g(y)/(1 - G(y)) \), and for \( y > 0 \)

\[
(3.2) \quad F_-(x, y, t + \Delta t) = (1 - g^*(y - \Delta t) \Delta t) F_-(x + \Delta t, y - \Delta t, t) + o(\Delta t).
\]
The function \( F_+ \) has a jump discontinuity along the line \( x = t \) of saltus \( e^{-\lambda t} \), but otherwise assume it is continuously differentiable with respect to \( x \) and \( t \) for \( x, t > 0 \). The function \( F_- \) has a jump discontinuity along the \((y, t)\) - plane (i.e., for \( x = 0 \)) and along the plane \( y = (t - x)/2 \) which is created by the discontinuity of \( F_+ \) along the line \( x = t \). Except on the latter plane assume \( F_- \) is continuously differentiable with respect to \( x, y, t \) for \( x > 0, 0 < y < t \).

Passage to the limit \((\Delta t \to 0)\) in (3.1) and (3.2) yields

\[
(3.3) \quad \frac{\partial F_+(x, t)}{\partial t} = -\frac{\partial F_+(x, t)}{\partial x} - \lambda F_+(x, t) + \int_0^\infty F_-(x, y, t) g^*(y)dy,
\]

for \( x, t > 0, x \neq t \), and

\[
(3.4) \quad \frac{\partial F_-(x, y, t)}{\partial t} = -\frac{\partial F_-(x, y, t)}{\partial y} + \frac{\partial F_-(x, y, t)}{\partial x} - g^*(y) F_-(x, y, t),
\]

for \( x > 0, 0 < y < t, y \neq (t - x)/2 \). The boundary conditions are

\[
(3.5) \quad F_-(x, 0, t) = \lambda F_+(x, t),
\]

\[
(3.6) \quad F_+(x, 0) = 1 \text{ for all } x,
\]

and

\[
(3.7) \quad F_-(x, y, 0) = 0 \text{ for all } x, y.
\]
Solution of the linear, first-order partial differential equation (3.4) in the two regions separated by the plane \( y = (t - x)/2 \) yields the relation

\[
F_- (x, y, t) = \lambda (1 - G(y)) F_+ (x + y, t - y).
\]

Substitution of this expression for \( F_- \) into (3.3) produces the following integro-differential equation:

\[
\frac{\partial F_+ (x, t)}{\partial t} = - \frac{\partial F_+ (x, t)}{\partial x} - \lambda F_+ (x, t) + \lambda \int_0^t F_+ (x + y, t - y) g(y) \, dy,
\]

for \( x, t > 0, x \neq t \). The transform of (3.9) with respect to \( t \) is the integro-differential equation

\[
\frac{\partial \tilde{F}_+ (x, u)}{\partial u} - F_+ (x, 0) = - \frac{\partial}{\partial x} \tilde{F}_+ (x, u) - \lambda \tilde{F}_+ (x, u) + \lambda \int_0^\infty \tilde{F}_+ (x + y, u) e^{-uy} g(y) \, dy.
\]

A term \( e^{-ux} (F_+ (x, x -) - F_+ (x, x +)) \) created by the discontinuity at \( t = x \) cancels from both sides. Homogeneous integro-differential equations of the type (3.10) (i.e., with \( F_+ (x, 0) = 0 \)) have been studied by Karlin and Szegö [10]. By a trivial modification of their argument it can be easily established that the only solution to the homogeneous equation

\[
\frac{\partial \tilde{F}_+ (x, u)}{\partial u} - (\lambda + u) \tilde{F}_+ (x, u) + \lambda \int_0^\infty \tilde{F}_+ (x + y, u) e^{-uy} g(y) \, dy,
\]
which tends to a finite limit as $x \to \infty$ is the exponential solution $\exp(-\alpha(u)x)$. For fixed $u$ the constant $\alpha(u)$ is the unique positive root of the equation

$$ (3.12) \quad 0 = \alpha - (\lambda + u) + \lambda \mathcal{L}(\alpha + u), $$

where $\mathcal{L}$ is the Laplace transform of $g$. To obtain the general solution to (3.10) all that is required is a particular solution to the non-homogeneous equation. In the special case where $F_+(x, u)$ satisfies the boundary condition (3.6) the constant (for fixed $u$)

$$ (3.13) \quad \beta(u) = \frac{1}{u + \lambda(1 - \mathcal{L}(u))} $$

furnishes a particular solution. Consequently, $\mathcal{F}_+(x, u) = \beta(u) + \gamma(u) \exp(-\alpha(u)x)$ is the general solution to (3.10) subject to (3.6). The other boundary conditions (3.5) and (3.7) imply that $\mathcal{F}_+(0, u) = 0$ for all $u$. This determines $\gamma(u)$ to be $\gamma(u) = -\beta(u)$ so

$$ (3.14) \quad \mathcal{F}_+(x, u) = \frac{1 - e^{-\alpha(u)x}}{u + \lambda(1 - \mathcal{L}(u))}. $$

From (3.8) and (3.14)

$$ (3.15) \quad \mathcal{F}_-(x, y, u) = \lambda(1 - G(y)) e^{-uy} \left[ \frac{1 - e^{-\alpha(u)(x + y)}}{u + \lambda(1 - \mathcal{L}(u))} \right]. $$
If $\lambda/\mu > 1$ where $\mu = 1/E(\eta)$, the process will have a limiting distribution as $t \to \infty$ since the pull of the process is towards the zero barrier. This limiting distribution is easily calculated from (3.14) and (3.15) by Karamata's tauberian theorem:

$$F_-(x, \infty) = \lim_{u \to 0} u \tilde{F}_-(x, u) = \frac{1-e^{-\frac{\alpha_0 x}{\lambda/\mu}}}{1 + \lambda/\mu},$$

$$F_-(x, y, \infty) = \lim_{u \to 0} u \tilde{F}_-(x, y, u) = \lambda(1 - G(y)) \left[ \frac{1-e^{-\frac{\alpha_0 (x+y)}{\lambda/\mu}}}{1 + \lambda/\mu} \right],$$

where $\alpha_0$ is the root of the equation

$$\alpha_0 = \lambda(1 - G(\alpha_0)).$$

The distribution of the length of a non-empty period was not needed in obtaining (3.14) and (3.15) as it was in the previous section, but it is still of interest per se. The same arguments apply to establish that $T = 2T^*$ where $T$ is the total length and $T^*$ is the amount of time spent decreasing. The distribution of $T^*$ is the same as the distribution of a busy period for the queue GI/M/1, and the Laplace-Stieltjes transform $\tilde{T}^*(s)$ of this distribution has been characterized by B. W. Conolly [3]. Stated in terms of $\tilde{T}(s)$ Conolly's results show that

$$\tilde{T}(s) = \frac{\lambda(1 - \xi(2s))}{s + \lambda(1 - \xi(2s))},$$
where \( \xi(s) \) is the unique analytic solution to the functional equation

\[
(3.20) \quad f(s) = \tilde{G}(s + \lambda(1 - f(s))),
\]

subject to either of the conditions \( \lim_{s \to \infty} f(s) = 0 \) or \( |f(s)| < 1 \) for \( s > 0 \).

As in the previous section the limiting distributions for two related stochastic storage processes are expressed by (3.16) and (3.17). When renormalized (3.17) is the limiting distribution of a storage process with shot increases and linear output while (3.16) renormalized is the limiting distribution for linear input and shot discharges. The first process with shot inputs corresponds to the waiting time process in a queue with exponential service and its limiting distribution has already been obtained by D. G. Kendall [12].
4. F/M/K and M/G/K.

It is immaterial whether the discussion centers on F/M/K or M/G/K since the incorporation of a zero type barrier at K makes the one process an upside down image of the other. Any path function for F/M/K becomes a path function for M/F/K when 0 and K are identified as K and 0, respectively, i.e., the state spaces are related by $S_{F/M/K} = K - S_{M/F/K}$. For simplicity the discussion will select the process F/M/K.

This storage process represents a finite or limited capacity storage system such as a dam. It is at the same time more realistic for certain applications and far more difficult to handle mathematically. The only result that has been obtained is the limiting distribution for the storage level which will exist regardless of the relative sizes of the mean values of $\xi$ and $\eta$.

Let $X(t)$ remain at zero initially for the length of time $\eta^{-1}$. Let $F_{-}(x, t) = P(X(t) \leq x, X(t) \text{ decreasing})$, and $F_{+}(x, y, t) \ dy = P(X(t) \leq x, y \leq Y(t) \leq y + dy, X(t) \text{ increasing})$ where the random variable $Y(t)$ is defined as in Section 2.

In the limit as $t \to \infty$ the forward Kolomogorov equations become

$$(k.1) \quad 0 = \frac{\partial F_{-}(x, \infty)}{\partial x} - \mu F_{-}(x, \infty) + \int_{0}^{\infty} F_{+}(x, y, \infty) f^{*}(y) \ dy,$$

for $0 < x < K$, and
\[ (4.2) \quad 0 = \frac{\partial F_+(x, y, \infty)}{\partial y} - \frac{\partial F_+(x, y, \infty)}{\partial x} - f^*(y) F_+(x, y, \infty), \]

for \( 0 < y < x < K \). The functions \( F_- \) and \( F_+ \) have been assumed to be continuously differentiable in the regions indicated. The boundary conditions are

\[ (4.3) \quad F_+(x, 0, \infty) = \mu F_-(x, \infty), \]

\[ (4.4) \quad F_-(K, \infty) = \frac{\lambda}{\lambda + \mu}, \]

\[ (4.5) \quad F_+(K, y, \infty) = \frac{\lambda \mu}{\lambda + \mu} (1 - F(y)). \]

The last two boundary conditions are simply limiting occupation probabilities for an alternating sequence of random variables (see Takács [18], Fabens [4]).

The solution to the partial differential equation (4.2) subject to (4.3) is

\[ (4.6) \quad F_+(x, y, \infty) = \mu (1 - F(y)) F_-(x - y, \infty), \]

for \( 0 < y < x < K \). Substitution of (4.6) into (4.1) yields

\[ (4.7) \quad 0 = \frac{\partial F_-(x, \infty)}{\partial x} - \mu F_-(x, \infty) + \mu \int_0^x F_-(x - y, \infty) f(y) \, dy \]
for $0 < x < K$. This is the limiting form of the Takács integro-differential equation truncated at $x = K$ which was analyzed in [6]. Except for a multiplicative constant $F_-$ is given by the inversion of $[1 - (\mu/s)(1 - F(s))]^{-1}$ for $x$ in the range $[0, K]$. The boundary condition (4.4) determines $F_-$ to be

$$F_-(x, \infty) = C_K^{-1} \sum_{n=0}^{\infty} \left( \frac{\mu}{\lambda} \right)^n H(n)(x),$$

for $0 \leq x \leq K$ where

$$C_K = \frac{\lambda + \mu}{\lambda} \sum_{n=0}^{\infty} \left( \frac{\mu}{\lambda} \right)^n H(n)(K),$$

and $H$ is defined by (2.19). From (4.6)

$$F_+(x, y, \infty) = C_K^{-1} \mu(1 - F(y)) \sum_{n=0}^{\infty} \left( \frac{\mu}{\lambda} \right)^n H(n)(x - y)$$

for $0 \leq y \leq x < K$. For $x = K$ the value $F_+(K, y, \infty)$ is given by (4.5).

As in the two previous sections the limiting distributions for two related finite storage processes can be obtained from (4.5), (4.8), and (4.10). The limiting distribution for a process with jump inputs and linear output is given by (4.8) renormalized. This process was discussed in [6]. The process with linear input and jump discharges has a limiting distribution given by (4.5) and (4.10) normalized.
5. M/M/∞.

For this case \( F(t) = 1 - e^{-\lambda t}, \lambda > 0, \) and \( G(t) = 1 - e^{-\mu t}, \mu > 0, \ t > 0. \) When both the input and output distributions are exponential, it is possible to obtain explicitly the time dependent and limit distributions and to study the process in greater detail in terms of its asymptotic transient behavior and various absorption probabilities.

Consider \( F_+(x, t) \) when \( X(t) \) increases initially. The Laplace transform of \( F_+(x, t) \) with respect to \( t \) is given by (3.14) where \( \alpha(u) \) is the positive root of the equation

\[
(5.1) \quad 0 = \alpha(u) - (\lambda + \mu) + \frac{\lambda u}{\mu + \alpha(u) + u},
\]

i.e.,

\[
(5.2) \quad \tilde{F}_+(x, u) = \frac{1 - \exp[\{(-x(\lambda - \mu))/2\} - x \sqrt{(u + (\lambda + \mu)/2)^2 - \lambda \mu}]}{u + (\frac{\lambda u}{\mu + u})}.
\]

Rewrite (5.2) as

\[
(5.3) \quad \tilde{F}_+(x, u) = \frac{u + \mu}{u(u + \lambda + \mu)} \left\{ \frac{1 - \exp(-x(u + \lambda)) - \exp(-(\lambda - \mu)x/2)}{x(\exp \left[ - x \sqrt{(u + (\lambda + \mu)/2)^2 - \lambda \mu} \right] - \exp \left[ - x(u + (\lambda + \mu)/2) \right]} \right\}.
\]
Inversion of (5.3) term by term leads to the following explicit expression for $F_+(x, t)$:

\begin{equation}
F_+(x, t) = F_+(\infty, t) - e^{-x \lambda} F_+(\infty, t - x) - \int_0^{t - x} k(x, t - w) F_+(\infty, w) \, dw,
\end{equation}

for $0 \leq x < t$ where

\begin{equation}
k(x, t) = \begin{cases} 
\frac{x \sqrt{\mu} \, I_1((\lambda \mu)^{1/2}(t^2 - x^2)^{1/2})e^{-x(\lambda - \mu)/2 - t(\lambda + \mu)/2}}{(t^2 - x^2)^{1/2}} & , t > x, \\
0 & , t \leq x,
\end{cases}
\end{equation}

and where $F_+(\infty, t)$ is obtained from the inversion of the common factor in (5.3), i.e.,

\begin{equation}
F_+(\infty, t) = \frac{\mu}{\lambda + \mu} + \frac{\lambda e^{-(\lambda + \mu)t}}{\lambda + \mu}.
\end{equation}

The value of $F_-(x, y, t)$ can be obtained from (5.4) - (5.6) by means of (3.8).

If \( \lambda/\mu > 1 \), the storage process has a limiting distribution as $t \to \infty$. This limit is expressed in (3.16) - (3.18), and in the case $M/M/\infty$ it can be written explicitly:

\begin{equation}
F_+(x, \infty) = \frac{1 - e^{-(\lambda - \mu)x}}{1 + \lambda/\mu},
\end{equation}
(5.8) \[ F_-(x, y, \infty) = \frac{\lambda e^{-\lambda y}(1 - e^{-(\lambda - \mu)(x+y)})}{1 + \lambda/\mu} \].

The distribution \( F_-(x, \infty) = \int_0^\infty F_-(x, y, \infty) \, dy \) has a jump at the origin of saltus

(5.9) \[ F_-(0, \infty) = \frac{\lambda - \mu}{\lambda + \mu} \].

If \( \lambda/\mu \leq 1 \) the drift of the process is away from the zero barrier. The level of the storage system will tend to become infinitely large and no limiting distribution will exist as \( t \to \infty \). Just how the level drifts away from zero can be described by finding the asymptotic rate of growth of its moments and its limiting normalized distribution.

Consider first the case \( \lambda/\mu < 1 \). The Laplace transforms with respect to \( t \) of the expected storage levels can be obtained from (3.14) and (3.15):

(5.10) \[ \int_0^\infty e^{-ut} E_+(X(t)) \, dt = \frac{1}{\alpha(u)(u + \lambda(1 - \tilde{a}(u)))} \].

(5.11) \[ \int_0^\infty e^{-ut} E_-^-(X(t)) \, dt = \frac{1}{\alpha(u)(u + \lambda(1 - \tilde{a}(u)))} \cdot \frac{\alpha(u) - u}{u + \alpha(u)} \],

where \( E_\pm \) denotes the expectation with respect to \( F_\pm \). Analysis of the equation (3.12) reveals that as \( u \to 0 \)
\[ (5.12) \quad \alpha(u) \sim u \left( \frac{1}{\frac{1}{1} - \lambda/\mu} \right) , \]

and

\[ (5.13) \quad \frac{\alpha(u) - u}{\alpha(u) + u} \rightarrow \frac{\lambda}{\mu} . \]

Consequently, as \( u \rightarrow 0 \)

\[ (5.14) \quad \int_{0}^{\infty} e^{-ut} E_{+}(X(t)) \, dt \sim \frac{1}{u^2} \cdot \frac{1 - \lambda/\mu}{(1 + \lambda/\mu)^2} , \]

\[ (5.15) \quad \int_{0}^{\infty} e^{-ut} E_{-}(X(t)) \, dt \sim \frac{1}{u^2} \cdot \frac{\lambda}{\mu} \cdot \frac{1 - \lambda/\mu}{(1 + \lambda/\mu)^2} . \]

From (5.14) and (5.15) and Karamata's tauberian theorem

\[ (5.16) \quad \int_{0}^{t} E_{+}(X(\tau)) \, d\tau \sim \frac{(1 - \lambda/\mu)t^2}{2(1 + \lambda/\mu)^2} , \]

\[ (5.17) \quad \int_{0}^{t} E_{-}(X(\tau)) \, d\tau \sim \frac{(\lambda/\mu)(1 - \lambda/\mu) t^2}{2(1 + \lambda/\mu)^2} . \]

The analysis so far has not depended on the fact that \( G \) is negative exponential and consequently is valid for general \( G \). However, a proof that \( E_{+}(X(t)) \) and \( E_{-}(X(t)) \) are sufficiently smooth to allow
for the removal of the integral signs in (5.16) - (5.17) was obtained only when \( G \) is Markovian. In this case the argument follows from the manipulation of a system of differential inequalities. Thus, for negative exponential \( G \),

\[
E_+(X(t)) \sim \frac{(1 - \lambda/\mu) \, t}{(1 + \lambda/\mu)^2},
\]

\[
E_-(X(t)) \sim \frac{(\lambda/\mu)(1 - \lambda/\mu) \, t}{(1 + \lambda/\mu)^2}.
\]

Similar tauberian arguments based on the first and second order terms show that the conditional variance of \( X(t) \) given that \( X(t) \) is increasing (or decreasing) at time \( t \) grows asymptotically as

\[
\text{Var}_+(X(t)) \sim \text{Var}_-(X(t)) \sim \frac{8\lambda \mu t}{(\lambda + \mu)^3}.
\]

When \( \lambda/\mu = 1 \) the analogous asymptotic expressions are:

\[
E_+(X(t)) \sim E_-(X(t)) \sim \frac{t^{1/2}}{\sqrt{2\pi}} ,
\]

\[
\text{Var}_+(X(t)) \sim \text{Var}_-(X(t)) \sim \left( 1 - \frac{2}{\pi} \right) \frac{t}{\lambda} .
\]
Comparison of these results with those of Gupta [8] and Takács [18] show that the asymptotic growth of the storage process is identical to the diffusion process without the reflecting barrier at zero for the transient case, i.e., $\lambda/\mu < 1$. However, in the null recurrent case ($\lambda/\mu = 1$) the asymptotic behavior of the two processes differ. Asymptotically the bilateral diffusion process has a finite mean and its variance is given by (5.20) with $\lambda = \mu$ instead of (5.22).

In the transient case ($\lambda/\mu < 1$) the limiting normalized distribution is normal. This follows easily from the work of Takács [18] and the following characterization of the process which is due to S. Karlin:

\begin{equation}
(5.23) \quad X(t) = P(t) - N(t) + Z(t),
\end{equation}

where $P(t)$, $N(t)$, and $Z(t)$ are, respectively, the amounts of time in the interval $[0, t]$ the process spends moving positively, negatively, and at zero. Since $N(t) = t - P(t)$,

\begin{equation}
(5.24) \quad X(t) = 2P(t) - t + Z(t).
\end{equation}

The asymptotic distribution of $P(t)$ has been derived by Takács [18], and the contribution due to $Z(t)$ is eliminated in the limit since

\begin{equation}
(5.25) \quad P\left( \lim_{t \to \infty} \frac{Z(t)}{\sqrt{t}} = 0 \right) = 1.
\end{equation}
Summarily, as \( t \to \infty \)

\[
(5.26) \quad P \left\{ \frac{X(t) - t(\mu - \lambda) / (\mu + \lambda)}{\sqrt{\sigma_{\mu t} / (\mu + \lambda)^3}} \leq x \right\} \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} \, dy.
\]

Some absorption questions can also be answered in the case where \( F \) and \( G \) are both Markovian. In particular, suppose \( X(t) = x \) at time \( t \). What is the probability the process subsequently reaches the level \( R \) \(( R > x )\) before it reaches zero? If the storage system is finite and \( R = K \), this is the probability the system reaches full capacity before it empties. This probability and the transform of the distribution of the time to absorption can be obtained from a system of linear differential equations.

Let \( A_+(z; x, R) \) be the probability the process reaches level \( R \) at or before time \( z \) and before it reaches zero given that at time \( 0 \) it is at level \( x \) and increasing. Let \( A_-(z; x, R) \) be the corresponding probability conditioned on \( X(0) \) being decreasing. The quantities \( A_+(\infty; x, R) \) and \( A_-(\infty; x, R) \) are the absorption probabilities irrespective of the time of absorption.

Consideration of the possible changes in the process in an infinitesimal interval of time leads to the following system of linear partial differential equations:

\[
(5.27) \quad \frac{\partial A_+(z; x, R)}{\partial z} - \frac{\partial A_+(z; x, R)}{\partial x} = -\lambda A_+(z; x, R) + \lambda A_-(z; x, R),
\]
\begin{equation}
(5.28) \quad \frac{\partial A_- (z; x, R)}{\partial z} + \frac{\partial A_- (z; x, R)}{\partial x} = \mu A_+ (z; x, R) - \mu A_- (z; x, R),
\end{equation}

for \( 0 < x < R, \ R - x < z \). The boundary conditions are

\begin{equation}
(5.29) \quad A_- (z; 0, R) = 0, \quad z \geq R,
\end{equation}

\begin{equation}
(5.30) \quad A_+ (z; R, R) = 1, \quad z \geq 0,
\end{equation}

\begin{equation}
(5.31) \quad A_+ (R - x; x, R) = e^{-\lambda (R - x)}.
\end{equation}

The first two conditions are immediate and the third follows from the fact that if the process is increasing initially there is a discrete probability, \( \exp[-\lambda (R - x)] \), that no change occurs in time \( R - x \) so the level reaches \( R \) in this time.

Explicit solution of the system of equations (5.27) - (5.28) subject to (5.29) - (5.31) is complicated, but the Laplace-Stieltjes transforms of \( A_+ \) and \( A_- \) with respect to \( z \) are easily obtained from a corresponding system of ordinary differential equations. Let

\begin{equation}
(5.32) \quad \mathcal{A}_\pm (s, x, R) = \int_{(R-x)}^{\infty} e^{-sz} dA_\pm (z; x, R).
\end{equation}

Multiplication of (5.27) and (5.28) by \( e^{-sz} \) and integration with respect to \( z \) leads to the system of ordinary differential equations:
(5.33) \[ \frac{\partial \tilde{A}_+ (s, x, R)}{\partial x} = (\lambda + s) \tilde{A}_+ (s, x, R) - \lambda \tilde{A}_- (s, x, R), \]

(5.34) \[ \frac{\partial \tilde{A}_- (s, x, R)}{\partial x} = \mu \tilde{A}_+ (s, x, R) - (\mu + s) \tilde{A}_- (s, x, R). \]

The boundary condition (5.31) is used in obtaining (5.33); the conditions (5.29) and (5.30) transform to

(5.35) \[ \tilde{A}_- (s, 0, R) = 0, \]

(5.36) \[ \tilde{A}_+ (s, 0, R) = 1. \]

The solution to the equations (5.33) - (5.34) subject to (5.35) - (5.36) is

(5.37) \[ \tilde{A}_+ (s, x, R) = \frac{a_1(s) b_2(s) e^{r_1(s)x}}{a_1(s) b_2(s)} - \frac{a_1(s) b_1(s) e^{r_2(s)x}}{a_1(s) b_1(s)}, \]

(5.38) \[ \tilde{A}_- (s, x, R) = \frac{e^{r_1(s)x}}{b_1(s)} - \frac{e^{r_2(s)x}}{b_2(s)}, \]

where

(5.39) \[ r_1(s) = \lambda - \mu - \sqrt{\frac{(2s + \lambda + \mu)^2}{2} - 4
\lambda \mu}, \]
\[ r_2(s) = \frac{\lambda - \mu + \sqrt{(2s + \lambda + \mu)^2 - 4\lambda\mu}}{2}, \]

\[ a_1(s) = \frac{2\lambda}{\lambda - \mu - 2s - \sqrt{(2s + \lambda + \mu)^2 - 4\lambda\mu}}, \]

\[ b_1(s) = \frac{2\mu}{\lambda - \mu + 2s - \sqrt{(2s + \lambda + \mu)^2 - 4\lambda\mu}}, \]

\[ a_2(s) = \frac{2\lambda}{\lambda - \mu - 2s + \sqrt{(2s + \lambda + \mu)^2 - 4\lambda\mu}}, \]

\[ b_2(s) = \frac{2\mu}{\lambda - \mu + 2s + \sqrt{(2s + \lambda + \mu)^2 - 4\lambda\mu}}. \]

Although the transforms (5.37) - (5.38) are not easily inverted, they yield immediately the probabilities of ultimate absorption at \( R \) instead of at zero by evaluation for \( s = 0 \). If \( \lambda \neq \mu \),

\[ A_+(\infty; x, R) = \frac{1 - \frac{\lambda}{\mu} e^{(\lambda-\mu)x}}{1 - \frac{\lambda}{\mu} e^{(\lambda-\mu)R}}, \]

\[ A_-(\infty; x, R) = \frac{1 - e^{(\lambda-\mu)x}}{1 - \frac{\lambda}{\mu} e^{(\lambda-\mu)R}}, \]

and if \( \lambda = \mu \),

\[ A_+(\infty; x, R) = \frac{1 + \lambda x}{1 + \lambda R}, \]
(5.48) \[ A_-(\omega; x, R) = \frac{\lambda x}{1 + \lambda R}. \]

The absorption probabilities (5.47) - (5.48) were obtained by J. Lamperti by a different method and communicated to the author.

The distribution of the maximum value achieved by the process during a non-empty period is derivable from (5.45) and (5.47). The quantity \( 1 - A_+(\omega; 0+, R) \) is the probability that if \( X(t) \) increases from zero initially it never reaches the height \( R \) before it returns to zero. Hence, if \( \lambda \neq \mu \),

(5.49) \[ P\left\{ \max_{\text{NEP}} X(t) \leq m \right\} = \frac{\lambda}{\mu} \cdot \frac{1 - e^{(\lambda - \mu)m}}{1 - \frac{\lambda}{\mu} e^{(\lambda - \mu)m}}, \]

and if \( \lambda = \mu \),

(5.50) \[ P\left\{ \max_{\text{NEP}} X(t) \leq m \right\} = \frac{\lambda m}{1 + \lambda m}, \]

where \( \max_{\text{NEP}} \) denotes the maximum over a non-empty period.
REFERENCES


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