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TECHNICAL REPORT NO. 69
May 2, 1961

PREPARED UNDER CONTRACT Nonr-225(52)
(NR-342-022)
FOR
OFFICE OF NAVAL RESEARCH
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This work was sponsored by the Army, Navy and Air Force through the Joint Services Advisory Group for Research Groups in Applied Mathematics and Statistics by Contract Nonr-225(52) (NR-342-022)

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1. Introductory Discussion and Summary.

Let \( x = (x_1, x_2, \ldots, x_n) \) be a normal random vector with zero expectation vector and with a variance-covariance matrix which has 1 for its diagonal elements and \( \rho \) for its off-diagonal elements. Consider the quantity

\[
(1.1) \quad I_n(h;\rho) = (2\pi)^{\frac{n}{2}} \left[ 1 + (n-1)\rho \right]^{-\frac{1}{2}} \int_h^{\infty} \cdots \int_h^{\infty} e^{-\frac{1}{2}Q(x)} \, dx_1 \cdots dx_n,
\]

where

\[
Q(x) = \left[ (1 + (n-1)\rho)(1-\rho) \right]^{-1} \left[ (1 + (n-2)\rho) \sum x_i^2 - 2\rho \sum_{j>i} x_ix_j \right]
\]

(1.2)

\[
= (1-\rho)^{-1} \left[ \sum x_i^2 - \rho(1 + (n-1)\rho)^{-1} \left( \sum x_i^2 \right) \right].
\]

Thus \( I_n(h;\rho) \) is the probability that each of \( n \) normally distributed, equally correlated and standardized random variables with common correlation \( \rho \) shall not fall short of \( h \). Clearly \( 1 - I_n(h;\rho) \) is also the distribution function of the random variable \( \max_i x_i \), and this supplies one application (cf. [3]) of \( I_n(h;\rho) \). A second application relates to the familiar one-factor model in factor analysis for the special case of equal weights [8]. Another situation in which knowledge of the distribution of \( I_n(h;\rho) \) is important is in some models of test design in psychology. Other applications will arise or probably exist at present.
In a previous paper [8] (see also [8] for further references), $I_n(h;\rho)$ was expressed as the product of the density function at the cut-off point $h = (h, h, \ldots, h)$ and an infinite power series in $h$. In this paper it will be shown for $h > 0$ that $I_n(h;\rho)$ can be expressed asymptotically as the product of the density function at $h$ and an infinite series in negative powers of $h$. This result can be regarded as the generalization for $n > 1$ of the well-known asymptotic expansion of Mill's ratio

\[ (1.3) \int_x^\infty e^{-\frac{t^2}{2}} dt / e^{-\frac{x^2}{2}} \sim x^{-1}(1 - x^{-2} + 1.3x^{-4} - 1.3.5x^{-6} + \ldots)(x > 0). \]

2. The Asymptotic Development of $I_n(h;\rho)$.

Under the transformation

\[ y_i = [1 + (n - 1) \rho]^{-\frac{1}{2}} \sum_{j=1}^{n} b_{ij} x_j, \]

\[ y_i = (1 - \rho)^{-\frac{1}{2}} \sum_{j=1}^{n} b_{ij} x_j \quad (i = 2, 3, \ldots, n), \]

where $((b_{ij}))$, $i, j = 1, 2, \ldots, n$, is orthogonal with $b_{ij} = n^{-\frac{1}{2}}$ ($j = 1, 2, \ldots, n$), (1.1) reduces to

\[ (2.2) \quad I_n(h;\rho) = (2\pi)^{-\frac{1}{2}n} \int_R \cdots \int_R e^{-\frac{1}{2} \sum y_i^2} dy_1 \cdots dy_n \]

with $R$ defined by
\[ (2.3) \quad R: \left[ 1 + (n-1) \rho \right]^{\frac{1}{2}} \left[ n(1-\rho) \right]^{-\frac{1}{2}} \sum_{j=2}^{n} b_{ij} y_j \geq (1-\rho)^{-\frac{1}{2}} h \]
\[ (i = 1, 2, \ldots, n) \]

[8]. \( R \) is a polyhedral half-cone in \( \mathbb{R} \)-space with vertex at the point \( (r_0, 0, 0, \ldots, 0) \), where

\[ (2.4) \quad r_0 = \left[ n/(1 + (n-1) \rho) \right]^{\frac{1}{2}} h, \]

such that the angle between any two faces of the cone is arc \( \cos - \rho \); further the axis of the cone passes through the origin in \( \mathbb{R} \)-space.

\( I_n(h; \rho) \) is, then, the probability measure under an \( n \)-dimensional spherical normal distribution with unit standard deviation in any direction of a regular, symmetrically oriented polyhedral half-cone with common dihedral angle arc \( \cos - \rho \), and with vertex at a distance \( r_0 \) from the center of the distribution. Let \( P \) be any point within the cone distant \( r \) from the center of the distribution, \( \xi \) from the axis of the cone and \( x \) from the vertex of the cone in a direction parallel to the axis. The probability-mass of an infinitesimal element of volume \( d\tau \) at \( P \) is

\[ (2.5) \quad (2\pi)^{\frac{n}{2}} e^{-\frac{1}{2} r^2} d\tau = (2\pi)^{\frac{n}{2}} e^{-\frac{1}{2} (r_0 + x)^2} dx \quad (2\pi)^{\frac{n-1}{2}} e^{-\frac{1}{2} r^2} dS, \]

where \( dS \) is the measure of an infinitesimal element in the \( (n-1) \)-flat orthogonal to the axis of the cone and distant \( x \) from the vertex (cf. [5]). Consider the probability-mass in that portion of the cone (an infinitesimal
"slab" demarcated by two adjoining (n-1) flats orthogonal to the axis of the cone and distant \( x \) and \( x + dx \) from the vertex of the cone. It is easily shown that the intersection of the first of these two flats with the cone is a regular (n-1) dimensional simplex with centroid at the foot of the perpendicular from \( P \) to the axis of the cone and with edges of length

\[
\left[ \frac{2n(1 + (n-1) \rho)}{(1-\rho)} \right]^{\frac{1}{2}} x.
\]

Let \( K_n(\beta) \) denote the probability measure under an \( N \)-dimensional spherical normal distribution with unit standard deviation in any direction of a regular \( N \)-dimensional simplex with centroid at the center of the distribution and with edges of length \( \beta \). Then according to (2.5) the probability measure of the infinitesimal slab is

\[
(2.6) \quad (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}(r_o^2+x)^2} \cdot K_{n-1} \left[ \left( \frac{2n(1 + (n-1) \rho)}{1 - \rho} \right)^{\frac{1}{2}} x \right].
\]

consequently, the probability measure of the cone is

\[
I_n(h;\rho) = \int_{-\infty}^{\infty} (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}(r_o^2+x)^2} K_{n-1} \left[ \left( \frac{2n(1 + (n-1) \rho)}{1 - \rho} \right)^{\frac{1}{2}} x \right] dx
\]

\[
(2.7) \quad = (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}r_o^2} \int_{-\infty}^{\infty} e^{-r_o x} e^{-\frac{1}{2}x^2} K_{n-1}(\lambda x) \ dx,
\]

where
\[ \lambda = \lambda_n(\rho) \]

\[ = \left[ 2n(1 + (n-1)\rho)/(1-\rho) \right]^{1/2} \]

and \( r_0 \) is given by (2.4). Formula (2.7) which is of considerable intrinsic interest may be used also to develop the required asymptotic expansion of \( I_n(h;\rho) \) for \( h > 0 \).

The \( K \)-functions are closely related to Godwin's \( G \)-function [1], [2] introduced in connection with the distribution of the absolute mean deviation in normal samples, and some further statistical applications of the functions have been discussed in [4] and [5]. Clearly, \( K_N(x) \) is bounded by 1. Again, it has been shown elsewhere [7] that \( K_N(x) \) has a power series expansion with infinite radius of convergence. Consequently, Watson's lemma [9] (p. 236) may be used to obtain a valid asymptotic expansion for the integral in (2.7) by expanding \( \exp(-x^2/2)K_{n-1}(\lambda x) \) in its Taylor series at \( x = 0 \) and integrating term by term. In fact, let

\[ \psi_n(x) = \psi_n(x;\lambda) = e^{-x^2/2}K_{n-1}(\lambda x) = \sum_{i=0}^{\infty} c_{n-1,i} x^i/i! , \]

1 The center of the distribution is interior or exterior to the half-cone according as to whether it is within or without the half-cone, corresponding to the cases \( h > 0 \) and \( h < 0 \). The integral formula for \( I_n(h;\rho) \) in (2.7) is valid for all \( h \), but for the asymptotic expansion developed subsequently (equ. (2.22)) \( h > 0 \). (The case \( h < 0 \) is not likely to be of practical interest, while \( I_n(0;\rho) \) is known to be equal to the normed measure of a regular \( (n-1) \)-dimensional spherical simplex with common dihedral angle \( \arccos \rho \). The reader is referred to [9] where tables of each normed measure are provided for \( n = 1(1)51 - 1 \) and \( \rho = 1/1, i = 1(1)12 \).)
where the $c_{n-1,i}$ are functions of $\lambda$ (and therefore of $\rho$). Then (2.7) gives with the aid of Watson's lemma,

\[(2.10) \quad I_{\lambda} (h; \rho) \sim (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}r_0^2} \sum_{i=0}^{\infty} c_{n-1,i} r_0^{i+1} \]

This is the required formula. It should be noted that the probability density in the original distribution at the point $(h, h, \ldots, h)$ is

\[(2.11) \quad (2\pi)^{-\frac{1}{2}} \left[ 1 + (n-1) \rho \right]^{-\frac{1}{2}} \left[ 1 - (1-\rho) \right]^{-\frac{1}{2}(n-1)} e^{-\frac{1}{2}r_0^2} \]

thereby justifying the assertion at the end of the introductory Section.

It now remains to determine the coefficients $c_{n-1,i}$ in (2.10). On differentiating (2.9) $j$ times at $x = 0$ we obtain after some simplification

\[c_{n-1+2k} = \psi_n^{(n-1+2k)} (0)\]

\[(2.12) \quad = \sum_{s=0}^{k} (-\frac{1}{2})^{k-s} \frac{(n-1+2k)!}{(k-s)!} \lambda^{n-1+2s} a_{n-1,n-1+2s},\]

\[c_{n-1,m} = 0 \quad (m = 0, 1, 2, \ldots, n - 2),\]

where $\psi_n^{(n-1+2k)} (0)$ is the $(n - 1 + 2k)$th derivative of $\psi_n (x)$ at $x = 0$ and the $a$'s are defined by

\[K_N (x) = \sum_{j=0}^{\infty} a_{N,j} x^j \quad (N = 0, 1, 2, \ldots)\]
\[ a_{N,j} = k_N^j(0)/j! \]. In the derivation of (2.12) use has been made of the fact that

\begin{align*}
  a_{N,j} &= 0 \quad (j = 0, 1, 2, \ldots, N - 1), \\
  a_{N,N+2r+1} &= 0 \quad (r = 0, 1, 2, \ldots).
\end{align*}

(2.13)

Formula (2.13) in its turn derives by induction from the following recursion relationship between the \( a \)'s proved elsewhere [7]:

\[ a_{N,s} = (2s)^{-1} \left[ \left( N + 1 \right)/\left( Nn \right) \right]^{1/2(s-1)/2} \sum_{q=0}^{\left[ (s-1)/2 \right]} \left[ -4N(N + 1) \right]^{-q} a_{N-1,s-1-2q}/q! \]

\[ (s = 1, 2, \ldots) \],

\[ \left[ (s-1)/2 \right] \text{ denoting, as usual, the integral part of } (s-1)/2 \]. Though (2.14) may be exploited to derive explicit expressions for the non-negative \( a \)'s these are more easily obtained recursively by repeated application of (2.14) on noting that

\begin{align*}
  a_{0,j} &= 0 \quad (j = 1, 2, \ldots), \\
  &= 1 \quad (j = 0).
\end{align*}

(2.15)

This yields for the first three non-negative \( a_{n-1,j} \),

\[ a_{n-1,n-1} = \frac{1}{n^2} \frac{n^2}{\left[ 2 \right]^{n-1} \frac{1}{2^{n-1}} \frac{1}{(n - 1)!}} \],

(2.16)
\( (2.17) \quad a_{n-1,n+1} = -\frac{\frac{1}{2}}{2^{n-1} \pi \frac{1}{2}(n-1)} \frac{1}{4(n+1)!} \)

\( (2.18) \quad a_{n-1,n+3} = \frac{\frac{1}{2}}{2^{n-1} \pi \frac{1}{2}(n-1)} \frac{(n-1)(n^2 + 7n - 6)}{32n} \frac{1}{(n+3)!} \)

((2.14) shows that the non-negative \( a \)'s oscillate in sign).

On applying (2.16), (2.17) and (2.18) in (2.12), the first three non-negative \( c \)'s are obtained:

\( (2.19) \quad c_{n-1,n-1} = (n-1)! \lambda^{n-1} a_{n-1,n-1} \)

\[ \frac{1}{2} \quad 2^{-n-1} - \frac{1}{2} \lambda^{n-1} \lambda^{n-1} \]

\( (2.20) \quad c_{n-1,n+1} = (n+1)! \left\{ -\frac{1}{2} \lambda^{n-1} a_{n-1,n-1} + \lambda^{n+1} a_{n-1,n+1} \right\} \)

\[ \frac{1}{2} \quad 2^{-n} - \frac{1}{2} \lambda^{n-1} \left\{ \frac{1}{2} n(n+1) \lambda^{n-1} + \frac{1}{4} \lambda^{n+1} \right\} \]
\[ c_{n-1,n+3} = (n+3)! \left( \frac{1}{2} a_{n-1,n-1} - \frac{1}{2} a_{n-1,n+1} + \lambda^{n+3} a_{n-1,n+3} \right) \]

\[(2.21) \quad = \frac{1}{n^2} 2^{-(n-1)} \pi - \frac{1}{2(n-1)} \left( \frac{1}{3} n(n+1)(n+2)(n+3) \lambda^{n+1} + \frac{1}{12} (n+2)(n+3) \lambda^{n+1} \right. \]
\[\left. + \frac{1}{32} \frac{(n-1)(n^2 + 7n - 6)}{n} \right) \lambda^{n+3} \]

Thus from (2.10),

\[ I_n(h; \rho) \sim (2\pi)^{-1/2} \left[ \frac{1}{2} \frac{1}{\pi} e^{-\frac{1}{2} \rho^2} \left( c_{n-1,n-1} r_0^{-(n+1)} + c_{n-1,n+1} r_0^{-(n+3)} + c_{n-1,n+3} r_0^{-(n+5)} + \ldots \right) \right], \]

(2.22)

where the first three coefficients in the asymptotic expansion are given by (2.19), (2.20) and (2.21) (further coefficients may be obtained in the manner shown). A slightly more convenient form of (2.22) is

\[ I_n(h; \rho) \sim \frac{1}{n^2} \pi \frac{1}{2} e^{-\frac{1}{2} \rho^2} \left( t/r_o \right)^{n-1} r_o^{-1} \]

\[(2.23) \quad \times \left[ 1 - \left( \frac{1}{2(n+2)} + t^2 \right) r_o^{-2} \right. \]
\[\left. + \left( \frac{1}{6(n+1)} + \frac{1}{2(n+2)} \right) t^2 + \frac{1}{2(n-1)}(n^2 + 7n - 6) n^{1/2} t^4 r_o^{-4} + \ldots \right], \]

where

\[ t \equiv t_n(\rho) = \lambda/2 \]

\[(2.24) \quad = \left[ n(1 + (n-1)/2(1 - \rho)^{1/2} \right] \]
and \((n)_m\) denotes \(n(n+1) \cdots (n+m-1)\). It will be noted that the present asymptotic expansion is particularly suitable for large \(r_0\) (i.e., the cut-off point is not near the center of the distribution) and algebraically small \(\rho\).

Finally, observe that for \(n = 1\) \((2.22)\) reduces to \((1.3)\), since \(\psi_0(x) = \exp(-x^2/2)\) and

\[
(2.25) \quad c_{0,2j} = (-\frac{1}{2})^j (2j)! / j!.
\]

(The polyhedral half-cone is here the interval \((h, \infty)\).) For \(n = 2\), \((2.22)\) reduces to

\[
(2.26) \quad I_2(h; \rho) \sim \pi^{-1} e^{-\frac{1}{2}t^2} e^{\frac{1}{2}r_0^2} \left[ 1 + \frac{15+10t^2+3t^4}{r_0^4} - \cdots \right].
\]

This agrees with a formula obtained previously [6] for the probability measure, \(W(h; \theta)\), under a standardized circular normal distribution of a sector of angle \(\alpha\), vertex at a distance \(h\) from the center of the distribution and with one arm of the sector passing through the latter point. The relationship between \(I_2\) and \(W\) is

\[
(2.27) \quad I_2(h; \rho) = 2W(h; \theta/2)
\]

where \(\theta = 2 \arctan t = 2 \arctan((1+\rho)/(1-\rho))^{\frac{1}{2}}\). It has been shown in [6] that the bivariate normal integral for arbitrary cut-off point may be expressed in terms of the difference of two \(W\)-functions (and therefore of two \(I_2\)-functions).
3. The Accuracy of the Asymptotic Expansion.

In this section we obtain an upper bound to the error induced by taking the first \( m \) terms of the asymptotic expansion as an approximation to \( I_n(h; \rho) \). In particular, this allows a weaker upper bound to be obtained, to the effect that the above error is numerically not greater than the \((m+1)\)th term of the expansion for all \( h \).

Let \( \phi \) be the angle between the axis of the half-cone and the line joining any point \( P \) and the vertex of the cone. Then (using the notation of Section 2)

\[
r^2 = r_o^2 + \xi^2 + 2r_o \xi \cos \phi ,
\]

and the probability-mass of an infinitesimal volume-element of content \( d\tau \) as \( P \) is

\[
(2\pi)^{-\frac{1}{2n}} \exp\left[-\frac{1}{2}r^2\right] d\tau = (2\pi)^{-\frac{1}{2n}} \cdot \exp\left[-\frac{1}{2}(r_o^2 + \xi^2 + 2r_o \xi \cos \phi)\right] \xi^{n-1} d\xi d\omega ,
\]

(3.1)

where \( d\omega \) is the solid angle subtended at the center of the distribution by the volume-element (or, equivalently, the surface-content of an infinitesimal element on the surface of a unit sphere whose center coincides with the center of the distribution). Thus the probability-mass of the half-cone is

\[
(3.2) \quad I_n(h; \rho) = (2\pi)^{-\frac{1}{2n}} e^{-\frac{1}{2}r^2} \int_0^\infty \int_0^{2\pi} e^{-(r_o \cos \phi)\xi} \xi^{n-1} e^{-\frac{1}{2}\xi^2} d\xi d\omega ,
\]

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where \( \Omega \) is the \((n-1)\)-dimensional regular spherical simplex (with common dihedral angle \( \arccos \rho \)) formed by the intersection of the half-cone and the surface of the unit sphere. Again, if

\[
G_{n-1}(\xi) = \xi^{n-1} e^{-\frac{\xi^2}{2}}
\]

then the derivatives of \( G_{n-1}(\xi) \) at the origin, \( G_{n-1}^{(q)}(0) \), are given by

\[
G_{n-1}^{(n-1+2i)}(0) = (-1)^i \frac{(n-1+2i)!}{2^i i!} \quad (i = 0, 1, 2, \ldots)
\]

with all other derivatives vanishing. Therefore, repeated integration by parts yields

\[
(3.3) \quad \int_0^\infty e^{-\left(r_o \cos \phi\right)\xi} G_{n-1}(\xi) d\xi = \sum_{i=0}^{m-1} (-1)^i \frac{(n-1+2i)!}{2^i i!} \frac{1}{(r_o \cos \phi)^{n+2i}} + R_m(r_o \cos \phi),
\]

where

\[
R_m(r_o \cos \phi) = (r_o \cos \phi)^{-n+2m-2} \int_0^\infty e^{-\left(r_o \cos \phi\right)\xi} G_{n-1}^{(n+2m-2)}(\xi) d\xi
\]

\[
(3.4)
\]

\[
= (r_o \cos \phi)^{-n+2m-1} \int_0^\infty e^{-\left(r_o \cos \phi\right)\xi} G_{n-1}^{(n+2m-1)}(\xi) d\xi
\]

after a further single integration by parts. On using (3.3) and (3.4) in (3.2),

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\[ I_n(h; \rho) = (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}r_o^2} \left\{ \sum_{i=0}^{m-1} \frac{(-1)^i (n-1+2i)!}{2^i i!} \alpha_{n,i} r_o^{-(n+2i)} \right\} + \int_{\Omega} R_m(r_o \cos \theta) \, d\omega \right\}, \]

(3.5)

where

\[ \alpha_{n,i} = \int_{\Omega} \sec^{n+2i} \theta \, d\omega. \]

(3.6)

In (3.5), the error after \( m \) terms is

\[ E_m = (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}r_o^2} \int_{\Omega} R_m(r_o \cos \theta) \, d\omega. \]

(3.7)

An upper bound to \( |E_m| \) can be obtained from an upper bound to \( R_m(r_o \cos \theta) \) in (3.4). The latter upper bound is itself obtained by deriving first an upper bound to \( \xi^{(n+2m-1)} \) for \( \xi \geq 0 \). If, then,

\[ |\xi^{(n+2m-1)}| \leq A_{n-1,2m}, \]

(3.8)

(3.4) gives for \( r_o > 0 \)

\[ |R_m(r_o \cos \theta)| < A_{n-1,2m} (r_o \cos \theta)^{-(n+2m)}, \]

(3.9)

whence by (3.7)
\[ |E_m| \leq (2\pi)^{\frac{1}{2}n} e^{-\frac{1}{2}r_o^2} A_{n-1,2m} \int_\Omega (r_o \cos \phi)^{-(n+2m)} d\omega \]

\[ = A_{n-1,2m} (2\pi)^{\frac{1}{2}n} e^{-\frac{1}{2}r_o^2} \alpha_{n,m} r_o^{-(n+2m)}, \]

which is proportional to the \((m+1)\)th term of the series

\[ \sum_{i=0}^{\infty} (-1)^i \frac{(n-1+2i)!}{2^{i+1} i!} \alpha_{n,i} r_o^{-(n+2i)}. \]

Consequently, (3.11) is a valid asymptotic expansion when \(r_o > 0\) of \(I_n(h;\rho)\). Moreover, the series (3.11) must be identical with the series (2.22), since a given function determines uniquely (if at all) a series of the form \(\sum c_p/r_o^p\), so that (3.10) provides an upper bound to the error in using (2.22).

We now proceed to determine a value\(^2/\) for \(A_{n-1,2m}\). Let

\[ \xi^{n-1} = \beta_{n-1,0} H_0(\xi) + \beta_{n-1,1} H_1(\xi) + \cdots + \beta_{n-1,n-1} H_{n-1}(\xi), \]

where \(H_j(\xi)\) are the Tchebycheff-Hemite polynomials orthogonal to the weight function \(\exp(-\xi^2/2)\) and normalized so that the coefficient of \(\xi^j\) in \(H_j(\xi)\) is 1. On multiplying (3.12) by \(H_j(\xi) \exp(-\xi^2/2)\), and integrating over the real line, we find

---

\(^2/\) That \(A_{n-1,2m} < \infty\) is evident from the fact that all derivatives of \(G_{n-1}(\xi)\) are products of polynomials in \(\xi\) and \(\exp(-\xi^2/2)\).
\[ \beta_{n-1,j} = \int_{-\infty}^{\infty} \xi^{n-1} H_j(\xi) e^{-\frac{1}{2} \xi^2} \int_{-\infty}^{\infty} H_j^2(\xi) e^{-\frac{1}{2} \xi^2} d\xi. \]

The value of the denominator in (3.13) is well-known to be \( \sqrt{2\pi} \cdot j! \). In order to evaluate the numerator, define

\[ \gamma_{n-1,j} = \int_{-\infty}^{\infty} \xi^{n-1} H_j(\xi) e^{-\frac{1}{2} \xi^2} d\xi. \]

Integration by parts gives the recursion relationship

\[ (3.14) \quad \gamma_{n-1,j} = (n-1) \gamma_{n-2,j-1}, \]

and on successive application of (3.14)

\[ \gamma_{n-1,j} = (n-1)(n-2) \cdots (n-j) \gamma_{n-1-j,0} \]

\[ = (n-1)(n-2) \cdots (n-j) \int_{-\infty}^{\infty} \xi^{n-1-j} e^{-\frac{1}{2} \xi^2} d\xi, \]

whence

\[ \gamma_{n-1,j} = (n-1)(n-2) \cdots (n-j) \frac{1}{2^{(n-j)/2}} \Gamma \left( \frac{1}{2}(n-j) \right) \quad \text{for even } n-1-j, \]

\[ (3.15) \quad = 0 \quad \text{for odd } n-1-j. \]
On substituting (3.15) in (3.13),

\[ \beta_{n-1,j} = \frac{(n-1)!}{2^{n-j}(n-1)!} \left[ \frac{1}{2}(n-1-j)! \right]! \]

\[ = 0 \quad \text{(n-1-j even)}, \]

\[ = 0 \quad \text{(n-1-j odd)}. \]

(3.16)

Reverting to (3.12),

\[ G_{n-1}(\xi) = \xi^{n-1} e^{-\frac{1}{2}\xi^2} \]

\[ = \sum_{j=0}^{n-1} \beta_{n-1,j} H_j(\xi) e^{-\frac{1}{2}\xi^2}, \]

and therefore

(3.17) \[ G_{n-1}^{(n+1+2m)}(\xi) = \sum_{j=0}^{n-1} \beta_{n-1,j} H_{j+2m}(\xi) e^{-\frac{1}{2}\xi^2}, \]

on recalling that

(3.18) \[ \frac{d^p}{d\xi^p} e^{-\frac{1}{2}\xi^2} = (-1)^p H_p(\xi) e^{-\frac{1}{2}\xi^2}. \]

An upper bound to \[ |H_{j+2m}(\xi)| e^{-\frac{1}{2}\xi^2} \] in (3.17) is readily deduced from the identity

\[ e^{-\frac{1}{2}\xi^2} = \int_{-\infty}^{\infty} e^{i\xi x} \left( \frac{\xi}{2\pi} \right)^p e^{-\frac{1}{2}\xi^2} dx \]
for real \( \xi \). Hence on applying (3.18),

\[
(-1)^p H_p(\xi) e^{-\frac{1}{2}\xi^2} = \int_{-\infty}^{\infty} (ix)^p e^{ix} (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}x^2} \, dx ,
\]

from which we obtain

\[
|H_p(\xi)| e^{-\frac{1}{2}\xi^2} \leq (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} |x|^p e^{-\frac{1}{2}x^2} \, dx
\]

(3.19)

\[
= \pi^{-\frac{1}{2}} 2^{\frac{1}{2}p} \Gamma \left[ \frac{1}{2}(p + 1) \right] .
\]

Thus from (3.16), (3.17) and (3.19),

(3.20)

\[
|G_{n-1}^{(n-1+2m)}(\xi)| \leq \pi^{-\frac{1}{2}} \sum_j \frac{(n-l)!}{2^{\frac{l}{2}(n-l-j)}} \frac{2^{m+j}}{\Gamma[m + \frac{1}{2}(j + 1)]} \frac{1}{[\frac{1}{2}(n-l-j)]!j!}
\]

\[
\sum_j \text{denoting summation over all non-negative integral } j \leq n-1 \text{ such that } n-1-j \text{ is even.}
\]

If \( n \) is odd, set \( j = 2i \) in (3.20). Then

\[
|G_{n-1}^{(n-1+2m)}(\xi)| \leq \pi^{-\frac{1}{2}} \frac{(n-1)/2}{\Gamma\left[\frac{1}{2}(n-1-2i)\right]} \frac{n-1!}{\left[\frac{1}{2}(n-1)\right]!(2i)!} \cdot 2^{m+i} \Gamma(m + i + \frac{1}{2}) ,
\]

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and, on using the duplication formula for the gamma function in the form

\[ -\frac{1}{2} \pi \Gamma(m + i + \frac{1}{2}) = (2m + 2i)! / \{(m + i)! \cdot 2^{2m+2i}\}, \]

the latter inequality simplifies to

\[ |G_{n-1}^{(n-1+2m)}(\xi)| \leq \frac{(n-1)!}{m + \frac{1}{2}(n-1)} \sum_{i=0}^{(n-1)/2} \frac{(2m + 2i)!}{(m + i)! \cdot (2i)!} \frac{1}{(\frac{n-1}{2} - i)!} \]

(3.21)

\[(n = 1, 3, \ldots) .\]

Similarly, if \( n \) is even, set \( j = 2i + 1 \) in (3.20). Then

\[ |G_{n-1}^{(n-1+2m)}(\xi)| \leq \pi \frac{1}{2} \frac{(n-2)/2}{\sum_{i=0}^{\frac{1}{2}(n-2-2i)}} \frac{(n-1)!}{2^{\frac{1}{2}(n-2-2i)} \left[\frac{1}{2}(n-2) - i\right]! \cdot (2i+1)!} \]

(3.22)

\[ \cdot \frac{m+i+\frac{1}{2}}{\Gamma(m + i + 1)}, \]

and, on using gamma duplication formula in the form

\[ -\frac{1}{2} \pi \Gamma(m + i + 1 + \frac{1}{2}) = (2m + 2i + 1)! / \{(m + i + \frac{3}{2})! \cdot 2^{2m+2i+1}\}, \]

the last inequality reduces to
\[ |a_{n-1}^{(n-1+2m)}(\xi)| \leq \frac{(n-1)!}{m+\frac{1}{2}(n-1)!} \sum_{i=0}^{\frac{n-2}{2}} \frac{(2m + 2i + 1)!}{\Gamma(m + i + \frac{3}{2})(2i + 1)!} . \]

\( (3.23) \)

\[ \cdot \frac{1}{(\frac{n-2}{2} - 1)!} \quad (n = 2, 4, \ldots) . \]

Formulae (3.21) and (3.23) provide the required inequalities in the sense that their right-hand members (refer to (3.8)) may be substituted for \( A_{n-1,2m} \) in (3.10) to supply the desired upper bound for the error after \( m \) terms. A weaker upper bound may be obtained by noting that in (3.21)

\[ \frac{(n-1)! (2m+2i)!}{(2i)!} = (n-1)! (2i+1)(2i+2) \cdots (2m+2i) \]

\[ \leq (n - 1 + 2m)! , \]

whence \(^3\) 

\[ |a_{n-1}^{(n-1+2m)}(\xi)| \leq \frac{(n - 1 + 2m)!}{2^m m!} \cdot \frac{\frac{1}{2}(n+1)}{\frac{1}{2}(n-1)} \]

\( (3.24) \)

\[ \leq \frac{(n - 1 + 2m)!}{2^m m!} \quad (n = 1, 3, \ldots) , \]

\(^3\) There are \((n + 1)/2\) terms in the series (3.21), and to obtain (3.24) the largest term of these \((n + 1)/2\) terms is substituted for each term.
since \( \frac{1}{2^{(n+1)}} \leq 1 \) for all odd \( n \). Similarly, for \( n \) even, observe that in (3.23)

\[
(n-1)! \cdot (2m+2i+1)! / (2i+1)! = (n-1)! \cdot (2i+2)(2i+3) \cdots (2m+2i+1)
\]

\[
\leq (n - 1 + 2m)!
\]

whence

\[
|g_{n-1}^{(n-1+2m)}(\xi)| \leq \frac{(n - 1 + 2m)!}{2^m \Gamma(m + 3/2)} \cdot \frac{1}{2^n(n-1)} \cdot \frac{n!}{2^m m!} \cdot \frac{1}{2^n(n-1)} \leq \frac{(n - 1 + 2m)!}{2^m m!}
\]

(3.25)

\[
(n = 2, 4, \ldots)
\]

since \( n/2^{(n+1)} < 1 \) for all even \( n \). An upper bound to \( |g_{n-1}^{(n-1+2m)}(\xi)| \) is thus \( (n - 1 + 2m)!/2^m m! \) for all \( n \). This upper bound may be substituted for \( A_{n-1,2m} \) in (3.10), thereby proving that the numerical error after \( n \) terms is less than the absolute value of the \( (m + 1) \)th term. It should be noted, however, that according to (3.25) this can be improved for even \( n \) by replacing \( A_{n-1,2m} \) by \( (n - 1 + 2m)! / (2^m \Gamma(m + 3/2)) \).
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