ASYMPTOTIC MULTIVARIATE OCCUPATION
TIME DISTRIBUTIONS FOR SEMI-MARKOV PROCESSES

By
RUPERT G. MILLER, JR.

TECHNICAL REPORT NO. 70
May 15, 1961

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0. Introduction and Summary.

The asymptotic normalized distribution of the time spent by a stochastic process in one of two possible states was first derived by L. Takács [9]. Let \( \{X(t), t \geq 0\} \) be a stochastic process such that at time \( t \), \( X(t) \in A \) or \( X(t) \in B \) where \( A \) and \( B \) are disjoint sets whose union is the whole state space. Let \( A_i(B_i), i = 1, 2, \ldots, \) be the time spent in state \( A(B) \) during its \( i \)th occupation. Under the assumption the alternating sequences \( \{A_i\} \) and \( \{B_i\} \) are independent and respectively identically distributed with finite second moments, Takács proved the characteristic integral

\[
S_A(t) = \int_0^t X_A(t) \, dt,
\]

where

\[
X_A(t) = \begin{cases} 
1 & \text{for } X(t) \in A, \\
0 & \text{for } X(t) \in B,
\end{cases}
\]

has a limiting normal distribution as \( t \to \infty \) and obtained the asymptotic moments.

With the aid of a theorem of F. Anscombe [1], A. Rényi [4] gave a much simpler proof of the same result. The general central limit problem for sequences \( \{A_i\} \) and \( \{B_i\} \) lacking finite second moments
was treated by Takács in [10], [11], and an extension in a different
direction was given by A. Renyi [5]. All of these results are in a sense
generalizations of the original work by H. Robbins [6] on random sums
of random variables.

In this paper asymptotic bivariate normality is established for
the cumulative occupation times of two states in a semi-Markov process
with countable state space and also for the cumulative sums of functions
defined on the occupation times. The asymptotic moments are given
explicitly for a general semi-Markov process with three possible states
and a semi-Markov process with countable state space in which
\[ F(i_1, j) = F(i), \text{ i.e.}, \; F(i_1, j) \text{ independent of } j. \]
These results are applied to the zero and one states in a simple \( \text{M}/\text{M}/1 \) queue.

Results of a similar type have been obtained recently by R. Pyke
[3] for the case in which \( F(i_1, j) = F_j, \; F(i_1, j) = F_i. \)

1. **Fundamental Theorem.**

The theorem given below is a straightforward extension of the
Anscombe-Renyi theorem to two dimensions. The reader should have no
difficulty in realizing that the extension could be carried to any
finite number of dimensions.

**Theorem 1:** Let \( (A_i, B_i), \; i = 1, 2, \ldots, \) be a sequence of independent,
identically distributed bivariate random variables with the following
finite moments: \( E(A_i) = E(B_i) = 0, \; E(A_i^2) = \sigma_i^2, \; E(B_i^2) = \sigma_i^2, \) and
\( E(A_i B_i) = \sigma_{i2}. \) Let \( A(t) \) and \( B(t) \) be positive integer-valued ran-
dom variables such that as \( t \to \infty, \; A(t)/t \to a \) and \( B(t)/t \to b \) in
probability where \( a \geq b > 0 \). Let

\[
\xi_t = \frac{1}{\sqrt{A(t)}} \sum_{i=1}^{A(t)} A_i \quad \text{and} \quad \eta_t = \frac{1}{\sqrt{B(t)}} \sum_{i=1}^{B(t)} B_i.
\]

Then as \( t \to \infty \),

\[
F_t(x, y) = P(\xi_t \leq x, \eta_t \leq y) \to \Phi(x, y; \sigma_1^2, \sigma_2^2, \sigma_{12} \sqrt{b/a}),
\]

where \( \Phi(\cdot, \cdot; \sigma_1^2, \sigma_2^2, \sigma_{12} \sqrt{b/a}) \) is the cumulative distribution function of a bivariate normal distribution with zero means, variances \( \sigma_1^2, \sigma_2^2 \) and covariance \( \sigma_{12} \sqrt{b/a} \). (Note that just as in the univariate case there is no assumption of independence between \( A(t), B(t), A_i, \) and \( B_i \) except between \( A_i \) and \( B_i \) in the special case \( \sigma_{12} = 0 \).)

**Proof:** Assume first that \( a > b \) and \( |\sigma_{12}| < \sigma_1 \sigma_2 \).

Let \( \epsilon > 0 \) be an arbitrarily chosen small number.

Since the probabilities of mutually exclusive events are additive,

\[
F_t(x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} P(\xi_t \leq x, \eta_t \leq y, A(t) = n, B(t) = m).
\]

By assumption \( A(t)/t \to a \) and \( B(t)/t \to b \) in probability as \( t \to \infty \), so there exists a constant \( t^* \) such that for \( t \geq t^* \),

\[
P\{|A(t) - at| < \epsilon t, |B(t) - bt| < \epsilon t\} \geq 1 - \epsilon.
\]

Hence, for \( t \geq t^* \),

3
\[
\epsilon \geq |F_t(x,y) - \\
\sum_{|n-at|<\epsilon t} \sum_{|m-bt|<\epsilon t} P(\xi_n \leq x, \eta_m \leq y, A(t) = n, B(t) = m)|,
\]

where \( \xi_n = \sum_{l=1}^{n} A_l/\sqrt{n} \) and \( \eta_m = \sum_{l=1}^{m} B_l/\sqrt{m} \).

For convenience let \( n_1 = \langle (a - \epsilon) t \rangle, n_2 = \langle (a + \epsilon) t \rangle, \)
\( m_1 = \langle (b - \epsilon) t \rangle, \) and \( m_2 = \langle (b + \epsilon) t \rangle \) where \( \langle r \rangle \) denotes the integral part of \( r \). Assume without loss of generality that \( \epsilon \) is chosen sufficiently small and \( t \) sufficiently large in order that \( m_1 < m_2 < n_1 < n_2 \). Then, for \( n_1 < n < n_2, m_1 < m < m_2, \)

\[
P(\xi_n \leq x, \eta_m \leq y, A(t) = n, B(t) = m) \]
\[
= P\left( \frac{1}{\sqrt{n}} \left( \sum_{l=1}^{m_1} A_l + \sum_{m_1+1}^{m_2-1} A_l + \sum_{m_2}^{n_1} A_l + \sum_{n_1+1}^{n} A_l \right) \leq x, \frac{1}{\sqrt{m}} \left( \sum_{l=1}^{m_1} B_l + \sum_{m_1+1}^{m_2-1} B_l \right) \right)
\]

\[
(1.4) \quad \leq y, A(t) = n, B(t) = m)
\]

\[
\leq P\left( \frac{1}{\sqrt{n}} \left( \sum_{l=1}^{m_1} A_l + \sum_{m_2}^{n_1} A_l \right) - \frac{1}{\sqrt{n}} (\rho_1 + \rho_2) \leq x, \frac{1}{\sqrt{m}} \sum_{l=1}^{m} B_l - \frac{1}{\sqrt{m}} \rho_2 \leq y, \right.
\]

\[
A(t) = n, B(t) = m),
\]

and

\[
P(\xi_n \leq x, \eta_m \leq y, A(t) = n, B(t) = m) \]

\[
(1.5) \quad \geq P\left( \frac{1}{\sqrt{n}} \left( \sum_{l=1}^{m_1} A_l + \sum_{m_2}^{n_1} A_l \right) + \frac{1}{\sqrt{n}} (\rho_1 + \rho_2) \leq x, \frac{1}{\sqrt{m}} \sum_{l=1}^{m} B_l + \frac{1}{\sqrt{m}} \rho_2 \leq y, \right.
\]

\[
A(t) = n, B(t) = m),
\]
where
\[ \rho_1 = \left| \sum_{m_1+1}^{m_2} A_i \right|, \quad \rho_2 = \max_{n_1 < n < n_2} \left| \sum_{n_1+1}^{n} A_i \right|, \quad \text{and} \quad \rho_3 = \max_{m_1 < m < m_2} \left| \sum_{m_1+1}^{m} B_1 \right|. \]

Chebyshev's inequality and Kolmogorov's inequality can be united through a Bonferroni inequality to give the following simultaneous bound on \( \rho_1, \rho_2, \) and \( \rho_3 \):
\[
P\left( \rho_1 \leq \sqrt[3]{\varepsilon \sqrt{m_1}}, \quad \rho_2 \leq \sqrt[3]{\varepsilon \sqrt{m_1}}, \quad \rho_3 \leq \sqrt[3]{\varepsilon \sqrt{m_1}} \right) \geq 1 - \left( \frac{m_2 - m_1 - 1}{m_1} \right) \left( \frac{\sigma_1^2 + \sigma_2^2}{\varepsilon^{2/3}} \right)
\]

(1.6)
\[
\left( \frac{n_2 - n_1 - 1}{m_1} \right) \frac{\sigma_1^2}{\varepsilon^{2/3}} \geq 1 - \frac{2(2\sigma_1^2 + \sigma_2^2)}{b - \varepsilon - 1/t} \sqrt[3]{\varepsilon}.
\]

Together the inequalities (1.3) - (1.6) yield
\[
F_{t}(x, y) \leq \sum_{|n-at| \leq \varepsilon t} \sum_{|m-bt| \leq \varepsilon t} P\left( \sum_{1}^{m_1} A_i + \sum_{m_2}^{n_1} A_i \leq \sqrt[n]{x} + 2 \sqrt[3]{\varepsilon \sqrt{m_1}}, \sum_{1}^{m_1} B_1 \leq \sqrt[m]{y} + 3 \sqrt[3]{\varepsilon \sqrt{m_1}}, A(t) = n, B(t) = m \right)
\]

(1.7)
\[
+ \varepsilon + \frac{2(2\sigma_1^2 + \sigma_2^2)}{b - \varepsilon - 1/t} \sqrt[3]{\varepsilon},
\]

and
\[ F_t(x,y) \geq \sum_{|n-at| < \epsilon t} \sum_{|m-bt| < \epsilon t} p \left( \frac{m_1}{m_2} \sum_{A_1 = 1}^{m_1} + \sum_{A_2}^{n_1} \leq \sqrt{n} x - 2 \frac{3}{\epsilon} \sqrt{m_1} \right), \]

\[ \sum_{l=1}^{m_1} B_i \leq \sqrt{m} y - \frac{3}{\epsilon} \sqrt{m_1}, \quad A(t) = n, B(t) = m \] - \epsilon

\[ - \frac{2(2\sigma_1^2 + \sigma_2^2)}{b - \epsilon - 1/t} \frac{3}{\epsilon}. \]

Since \( m_1 < n < n_2 \) and \( m_1 < m < m_2 \) in the summations of (1.7) and (1.8),

\[ F_t(x,y) \leq \sum_{|n-at| < \epsilon t} \sum_{|m-bt| < \epsilon t} \frac{1}{\sqrt{n_1}} p \left( \frac{1}{m_1} \sum_{A_1 = 1}^{m_1} + \sum_{A_2}^{n_1} \leq \sqrt{n_2} \frac{1}{n_1} x + 2 \frac{3}{\epsilon} \sqrt{\frac{1}{n_1}} \right), \]

\[ \frac{1}{\sqrt{m_1}} \sum_{l=1}^{m_1} B_i \leq \sqrt{\frac{m_2}{m_1}} y + \frac{3}{\epsilon}, \quad A(t) = n, B(t) = m \]

\[ + \epsilon + \frac{2(2\sigma_1^2 + \sigma_2^2)}{b - \epsilon - 1/t} \frac{3}{\epsilon}, \]

and

\[ F_t(x,y) \geq \sum_{|n-at| < \epsilon t} \sum_{|m-bt| < \epsilon t} p \left( \frac{1}{\sqrt{n_1}} \sum_{A_1 = 1}^{m_1} + \sum_{A_2}^{n_1} \leq x - 2 \frac{3}{\epsilon} \sqrt{\frac{m_1}{n_1}} \right), \]

\[ \frac{1}{\sqrt{m_1}} \sum_{l=1}^{m_1} B_i \leq y - \frac{3}{\epsilon}, \quad A(t) = n, B(t) = m \]

\[ - \epsilon - \frac{2(2\sigma_1^2 + \sigma_2^2)}{b - \epsilon - 1/t} \frac{3}{\epsilon}. \]
Since the total probability for $n, m$ outside the range of summation

\[ |n - at| < \epsilon t, \ |m - bt| < \epsilon t \] is less than $\epsilon$,

\[
F_t(x, y) \leq P\left\{ \frac{1}{\sqrt{n_1}} \left( \sum_{l=1}^{m_1} A_1 + \sum_{m_2} A_1 \right) \leq \sqrt{\frac{n_2}{n_1}} x + 2 \sqrt{\frac{m_1}{n_1}} \sqrt{\frac{1}{m_1}} \sum_{l=1}^{m_1} B_1 \leq \sqrt{\frac{m_2}{m_1}} y \right. \\
\left. + \frac{3}{2} \sqrt{\epsilon} \right\} + \epsilon + \frac{2(2\sigma_1^2 + \sigma_2^2)}{b - \epsilon - 1/t} \frac{3}{2} \sqrt{\epsilon},
\]

(1.11)

and

\[
F_t(x, y) \geq P\left\{ \frac{1}{\sqrt{n_1}} \left( \sum_{l=1}^{m_1} A_1 + \sum_{m_2} A_1 \right) \leq x - 2 \sqrt{\frac{m_1}{n_1}} \sqrt{\frac{1}{m_1}} \sum_{l=1}^{m_1} B_1 \leq y - \frac{3}{2} \sqrt{\epsilon} \right\} \]

(1.12)  

\[ - 2\epsilon - \frac{2(2\sigma_1^2 + \sigma_2^2)}{b - \epsilon - 1/t} \frac{3}{2} \sqrt{\epsilon} . \]

As $t \to \infty$, the limits of the right hand sides in (1.11) and (1.12) can be evaluated by the bivariate central limit theorem;

\[
\phi(x - 2 \sqrt{\frac{b - \epsilon}{a - \epsilon}}, y - \frac{3}{2} \sqrt{\epsilon}; \sigma_1^2 \frac{a - 3\epsilon}{a - \epsilon}, \sigma_2^2, \sigma_{12} \sqrt{\frac{b - \epsilon}{a - \epsilon}}, - 2\epsilon - \frac{2(2\sigma_1^2 + \sigma_2^2)}{b - \epsilon - 1/t} \frac{3}{2} \sqrt{\epsilon}
\]

\[
\leq \lim_{t \to \infty} F_t(x, y) \leq \phi\left( \frac{a + \epsilon}{a - \epsilon} x + 2 \sqrt{\frac{b + \epsilon}{a - \epsilon}}, \frac{b + \epsilon}{a - \epsilon}, y + \frac{3}{2} \sqrt{\epsilon}; \sigma_1^2 \left( \frac{a - 3\epsilon}{a - \epsilon} \right), \sigma_2^2, \sigma_{12} \sqrt{\frac{b - \epsilon}{a - \epsilon}} \right) + \epsilon + \frac{2(2\sigma_1^2 + \sigma_2^2)}{b - \epsilon} \frac{3}{2} \sqrt{\epsilon} .
\]

(1.13)

Since $\epsilon$ can be chosen arbitrarily small, the conclusion of the theorem follows from (1.13).

The cases in which $a = b$ and/or $|\sigma_{12}| = \sigma_1 \sigma_2$ can be proved similarly.

Let \( \{X(t), t \geq 0\} \) be a semi-Markov process whose state space consists of the positive integers. It will be assumed for simplicity that at time \( t = 0 \) the process has just entered state one. The behavior of the process for \( t > 0 \) is completely determined by \( P = (p_{ij}) \), the matrix of transition probabilities, and by \( F = (F_{ij}(\cdot)) \), the family of transition cumulative distribution functions. The restrictions on \( P \) and \( F \) are as follows:

\[
p_{ij} \geq 0, \quad \sum_j p_{ij} = 1, \quad p_{ii} = 0;
\]

\[
F_{ij}(0^{+}) = 0, \quad F_{ij}(+\infty) = 1.
\]

The interpretation of the matrices \( P \) and \( F \) is that when the process enters state \( i \) it moves to state \( j, j \neq i \), at the next transition with probability \( p_{ij} \) and prior to the transition it remains in state \( i \) for a random length of time whose probability distribution is \( F_{ij} \).

(For a fuller discussion of semi-Markov processes the reader is referred to [2], [7], [8].)

Let \( A_i \) be the length of time the process remains in state \( i \) at its \( i \)th visit to this state, \( B_i \) be the length of the \( i \)th visit to state 2, \( C_i \) be the length of the \( i \)th visit to state 3, etc. For fixed time \( t \) let the random variable \( A(t) \) be the number of visits to state 1 in the time interval \([0, t]\), \( B(t) \) be the number of visits to state 2, etc. Let
\[ \mu_{ij} = \int_0^\infty x dF_{ij}(x), \quad \mu_i = \sum_j p_{ij} \mu_{ij}, \]

\[ \mu_{ij}^{(2)} = \int_0^\infty x^2 dF_{ij}(x), \quad \sigma_{ij}^2 = \int_0^\infty (x - \mu_{ij})^2 dF_{ij}(x). \]

Let \( m_i \) be the mean recurrence time for state \( i \); \( m_i \) is assumed to be finite. For any square integrable functions \( f(\cdot) \) and \( g(\cdot) \) let

\[ \mu_{ij}(f) = \int_0^\infty f(x) dF_{ij}(x), \quad \mu_i(f) = \sum_j p_{ij} \mu_{ij}(f), \]

\[ \mu_{ij}^{(2)}(f) = \int_0^\infty f^2(x) dF_{ij}(x), \quad \sigma_{ij}^2(f) = \int_0^\infty (f(x) - \mu_{ij}(f))^2 dF_{ij}(x), \]

and

\[ \mu_{ij}(f, g) = \int_0^\infty f(x) g(x) dF_{ij}(x). \]

The asymptotic \( (t \to \infty) \) bivariate distribution of the random variables

\[
(2.1) \quad \left( \sum_1^t f(A_i) - \frac{t \mu_i(f)}{m_i}, \quad \sum_1^t g(A_i) - \frac{t \mu_i(g)}{m_i} \right)
\]

and

\[
(2.2) \quad \left( \sum_1^t f(B_i) - \frac{t \mu_2(f)}{m_2}, \quad \sum_1^t g(B_i) - \frac{t \mu_2(g)}{m_2} \right)
\]

will be shown to be a bivariate normal. If at time \( t \) the process is in state \( 1 \), the variable \( A_i \) for \( i = A(t) \) will customarily be interpreted as the occupation time up to \( t \), and similarly for the other state variables. Since the initial ordering of the states was entirely arbitrary,
there is no loss in generality in considering just the adjacent states 1 and 2.

The functions \( f(\cdot) \) and \( g(\cdot) \) which are primarily of interest are \( f(x) = x \) and \( g(x) = 1 \). In this case

\[
(2.3) \quad \sum_{1}^{A(t)} f(A_1) - \frac{t_{\mu_1}(f)}{m_1} = \sum_{1}^{A(t)} A_1 - \frac{t_{\mu_1}}{m_1}
\]

which is the centered cumulative occupation time for state 1, and

\[
(2.4) \quad \sum_{1}^{A(t)} g(A_1) - \frac{t_{\mu_1}(g)}{m_1} = A(t) - \frac{t}{m_1}
\]

which is the centered number of visits to state 1 in time \( t \).

The proof of the asymptotic bivariate normality of (2.1) and (2.2) follows easily from Theorem 1, but the evaluation of the asymptotic moments is more difficult and will be deferred until the next section.

For fixed time \( t \) let \( 0 = t_1 < t_2 < \cdots < t_{A(t)} \leq t \) be the times at which the process returns to the first state. Define the random variable \( \sum_{1}^{i} B_j \) to be the total length of time spent in state 2 between the \( i \)th and \((i + 1)\)st visit to state 1. The sum \( \sum_{1}^{i} B_j \) is a random sum of random variables - the number of variables in the sum is the number of visits to state 2 between the \( i \)th and \((i + 1)\)st return to state 1 and each variable \( B_j \) in the sum is the length of the \( j \)th sojourn in state 2 between the returns. Of course, \( \sum_{1}^{i} B_j \) may be zero.

Similar definitions apply to \( \sum_{1}^{i} C_j \), \( \sum_{1}^{i} D_j \), etc. With these definitions the \( A(t) \)th return to state 1 is equal to
(2.5) \[ t_A(t) = \sum_{i=1}^{A(t)-1} (A_1 + \sum \frac{i}{B_j} + \sum \frac{i}{C_j} + \cdots). \]

The variables entering into (2.1) and (2.2) can now be written in the following form:

(2.6) \[ S_{r(A)}^*(t) = \sum_{i} A_{i} \cdot f(A_i) - \frac{t \mu_1(f)}{m_1} \]

\[ = \sum_{i} \left[ f(A_i) - \frac{\mu_1(f)}{m_1} \right] \left( A_1 + \sum \frac{i}{B_j} + \sum \frac{i}{C_j} + \cdots \right) \]

\[ + \left[ f(A_{A(t)}) - \frac{\mu_1(f)}{m_1} \right] (t - t_{A(t)}) \]

(2.7) \[ S_{g(B)}^*(t) = \sum_{i} g(B_i) \cdot B_{i} - \frac{t \mu_2(g)}{m_2} \]

\[ = \sum_{i} \left[ \sum g(B_j) - \frac{\mu_2(g)}{m_2} \right] \left( A_1 + \sum \frac{i}{B_j} + \sum \frac{i}{C_j} + \cdots \right) \]

\[ + \left[ \sum A(t) g(B_j) - \frac{\mu_2(g)}{m_2} \right] (t - t_{A(t)}) \],

where \( \sum g(B_j) \) is defined analogously to \( \sum \frac{i}{B_j} \) and \( \sum A(t) g(B_j) \) is the sum of the \( g(B_j) \) for those visits occurring after time \( t_{A(t)} \) and before time \( t \). Define

(2.8) \[ X_i = f(A_i) - \frac{\mu_1(f)}{m_1} \left( A_1 + \sum \frac{i}{B_j} + \sum \frac{i}{C_j} + \cdots \right), \]
\begin{align}
(2.9) \quad Y_i &= g(A_i) - \frac{\mu_1(g)}{m_1} (A_i + \sum \gamma B_j + \sum \gamma C_j + \ldots), \text{ and} \\
(2.10) \quad Z_i &= \sum \gamma g(B_j) - \frac{\mu_2(g)}{m_2} (A_i + \sum \gamma B_j + \sum \gamma C_j + \ldots). 
\end{align}

Then, with the remainder terms defined appropriately,

\begin{align}
(2.11) \quad S^*_f(A)(t) &= \sum X_i + R_f(A)(t), \\
(2.12) \quad S^*_g(A)(t) &= \sum Y_i + R_g(A)(t), \text{ and} \\
(2.13) \quad S^*_g(B)(t) &= \sum Z_i + R_g(B)(t).
\end{align}

Since \((X_i, Y_i, Z_i)\) is independent of \((X_i', Y_i', Z_i')\) for \(i \neq i'\), by Theorem 1

\begin{align}
(2.14) \quad \left( \frac{1}{\sqrt{A(t)-1}} \sum X_i, \frac{1}{\sqrt{A(t)-1}} \sum Y_i \right)
\end{align}
is asymptotically normal with means \(E(X_i) = E(Y_i) = 0\), variances \(\text{var}(X_i), \text{var}(Y_i)\), and covariance \(\text{cov}(X_i, Y_i)\) since \(A(t)/t \to 1/m_1\) in probability as \(t \to \infty\) provided, of course, these variances and covariance exist. Similarly,

\begin{align}
(2.15) \quad \left( \frac{1}{\sqrt{A(t)-1}} \sum X_i, \frac{1}{\sqrt{A(t)-1}} \sum Z_i \right)
\end{align}
is asymptotically normal with zero means, variances \( \text{var}(X_1) \), \( \text{var}(Z_1) \), and covariance \( \text{cov}(X_1, Z_1) \). It follows from Lemma 3 of [4] that provided \( f \) and \( g \) are functions of bounded variation over all finite intervals \([0, t]\)

\[
(2.16) \quad \frac{R_{f(A)}(t)}{\sqrt{A(t)-1}}, \quad \frac{R_{g(A)}(t)}{\sqrt{A(t)-1}}, \quad \text{and} \quad \frac{R_{g(B)}(t)}{\sqrt{A(t)-1}}
\]

all converge in probability to zero as \( t \to \infty \). Finally, since \( \frac{A(t)}{(A(t)-1)} \) converges in probability to 1 as \( t \to \infty \), the bivariate analogue of Lemma 1 of [4] establishes that

\[
(2.17) \quad \frac{1}{\sqrt{A(t)}} \quad (S_{f(A)}^*(t), S_{g(A)}^*(t))
\]

is asymptotically normal with mean \((0,0)\) and variances-covariance \( \text{var}(X_1), \text{var}(Y_1), \text{and} \, \text{cov}(X_1, Y_1) \).

A similar argument establishes that as \( t \to \infty \),

\[
(2.18) \quad \frac{1}{\sqrt{A(t)}} \quad (S_{f(A)}^*(t), S_{g(B)}^*(t))
\]

is asymptotically normal with mean \((0,0)\) and variances-covariance \( \text{var}(X_1), \text{var}(Z_1), \text{and} \, \text{cov}(X_1, Z_1) \).

3. **Asymptotic Moments.**

To complete the specification of the asymptotic distribution of (2.17) and (2.18) it is necessary to determine \( \text{var}(X), \text{var}(Y), \text{var}(Z), \text{cov}(X, Y), \text{cov}(X, Z), \text{and} \, \text{cov}(Y, Z) \), where \( X, Y, \) and \( Z \) are generic
random variables representing (2.8), (2.9), and (2.10), respectively. Unfortunately, the extent of the algebraic manipulations prevents the calculation of these moments in terms of \( F = (\mu_{ij}) \) and the moments of \( F = (F_{ij}) \) for a general semi-Markov process. For the special cases of a general semi-Markov process with \( m = 3 \) states and of a semi-Markov process with \( F_{ij} = F_i \) and a finite or countable number of states these moments can be obtained explicitly.

3.1. General Semi-Markov Process; \( m = 3 \).

The second order moments mentioned above can be determined from general expressions for \( \mathbb{E}[\sum_i f(B_j)], \mathbb{E}[(\sum_i f(B_j))^2], \mathbb{E}[\sum_i f(B_j) \sum_i g(B_j)], \mathbb{E}[\sum_i f(B_j) \sum_i g(C_j)], \) and \( \mathbb{E}[f(A_i) \sum_i g(B_j)] \) where \( f \) and \( g \) are arbitrary functions with finite second moments. From elementary but tedious algebraic manipulations these expressions are:

\[
\mathbb{E}[\sum_i f(B_j)] = \left[ \frac{(p_{12} + p_{13}^2 p_{32}) p_{21}}{1 - p_{23} p_{32}} \right] \mu_{21}(f) \\
+ \left[ \frac{(p_{12} + p_{13} p_{32}) ((p_{23} p_{32}) p_{21} + p_{23} p_{31})}{(1 - p_{23} p_{32})^2} \right] \mu_{23}(f),
\]

(3.1)
\begin{align*}
\mathbb{E}(\sum \mathbb{I}_{f(B_j)})^2 &= \left[ \left( \frac{p_{12} + p_{13} p_{32}}{1 - p_{23} p_{32}} \right) p_{21} \right] \mu_{21}(f) \\
&+ \left[ \frac{2(p_{12} + p_{13} p_{32})(p_{23} p_{32})(p_{21} + p_{23} p_{31})}{(1 - p_{23} p_{32})^2} \right] \mu_{23}(f) \mu_{23}(g) \\
&+ \left[ \frac{2(p_{12} + p_{13} p_{32})(p_{23} p_{32}) p_{21}}{(1 - p_{23} p_{32})^2} \right] \mu_{21}(f) \mu_{23}(g) \\
&+ \left[ \frac{2(p_{12} + p_{13} p_{32})(p_{23} p_{32})}{1 - p_{23} p_{32}} \right] \mu_{21}(f) \mu_{21}(g),
\end{align*}

\begin{align*}
\mathbb{E}(\sum \mathbb{I}_{f(B_j)}) \sum \mathbb{I}_{g(B_j)} &= \left[ \left( \frac{p_{12} + p_{13} p_{32}}{1 - p_{23} p_{32}} \right) p_{21} + p_{23} p_{31} \right] \mu_{23}(f, g) \\
&+ \left[ \frac{2(p_{12} + p_{13} p_{32})(p_{23} p_{32})(p_{21} + p_{23} p_{31})}{(1 - p_{23} p_{32})^3} \right] \mu_{23}(f) \mu_{23}(g) \\
&+ \left[ \frac{(p_{12} + p_{13} p_{32})(p_{23} p_{32}) p_{21}}{(1 - p_{23} p_{32})^2} \right] (\mu_{21}(f) \mu_{23}(g) + \mu_{23}(f) \mu_{21}(g)) \\
&+ \left[ \frac{(p_{12} + p_{13} p_{32})}{1 - p_{23} p_{32}} \right] \mu_{21}(f, g),
\end{align*}
\begin{align}
E\left( \sum \mathbf{1}_f(B_j) \right) &= E\left( \sum \mathbf{1}_g(C_j) \right) = \left[ \frac{(p_{12} + p_{13} p_{32})(p_{23} p_{31})}{(1 - p_{23} p_{32})^2} \right] \mu_{23}(r) \mu_{31}(g) \\
&+ \left[ \frac{(p_{13} + p_{12} p_{23})(p_{32} p_{21})}{(1 - p_{23} p_{32})^2} \right] \mu_{21}(r) \mu_{32}(g) \\
&+ \left[ \frac{(p_{23} p_{32})(p_{12} + p_{13} p_{32}) + (p_{13} + p_{12} p_{23})}{(1 - p_{23} p_{32})^2} \right] \mu_{23}(r) \mu_{32}(g),
\end{align}

and

\begin{align}
E\left( \sum \mathbf{1}_g(B_j) \right) = E\left( \sum \mathbf{1}_f(C_j) \right) = \left[ \frac{p_{12}((p_{23} p_{32}) p_{21} + p_{23} p_{31})}{(1 - p_{23} p_{32})^2} \right] \mu_{12}(r) \mu_{23}(g) \\
&+ \left[ \frac{p_{12} p_{21}}{1 - p_{23} p_{32}} \right] \mu_{12}(r) \mu_{21}(g) \\
&+ \left[ \frac{(p_{13} p_{32})(p_{23} p_{32}) p_{21} + p_{23} p_{31}}{(1 - p_{23} p_{32})^2} \right] \mu_{13}(r) \mu_{23}(g) \\
&+ \left[ \frac{(p_{13} p_{32}) p_{21}}{1 - p_{23} p_{32}} \right] \mu_{13}(r) \mu_{21}(g).
\end{align}

3.2. Semi-Markov Process with \( F_{ij} = F_1 \); \( m \leq \infty \).

Since \( F_{ij} \) is independent of \( j \), the subscript \( j \) will be dropped in the symbols \( \mu_{ij}(r) \), \( \mu_{ij}^{(2)}(r) \), etc. Let \( I(\cdot) \) be the function \( I(x) \equiv 1, x \geq 0 \), and let
\begin{equation}
1_{N_2}^N = \mathbb{E}(\sum_1^1 I(B_j)), \quad 1_{N_2}^{(2)} = \mathbb{E}(\sum_1^1 I(B_j))^2),
\end{equation}

\begin{equation}
1_{N_3}^N = \mathbb{E}(\sum_1^1 I(C_j)), \quad 1_{N_3}^{(2)} = \mathbb{E}(\sum_1^1 I(C_j))^2),
\end{equation}

\begin{equation}
1_{N_{23}}^N = \mathbb{E}(\sum_1^1 I(B_j) \sum_1^1 I(C_j)), \text{ etc.}
\end{equation}

The quantity $1_{N_2}$ is the expected number of visits to state 2 between visits to state 1, etc. In terms of these expectations

\begin{equation}
(3.7) \quad \mathbb{E}(\sum_1^1 f(B_j)) = 1_{N_2} \cdot \mu_2(f),
\end{equation}

\begin{equation}
(3.8) \quad \mathbb{E}(\sum_1^1 f(B_j))^2 = 1_{N_2} \cdot \sigma_2^2(f) + 1_{N_2}^{(2)} \cdot (\mu_2(f))^2,
\end{equation}

\begin{equation}
(3.9) \quad \mathbb{E}(\sum_1^1 f(B_j) \sum_1^1 g(B_j)) = 1_{N_2} \cdot \mu_2(f, g)
\end{equation}

\begin{equation}
+ (1_{N_2}^{(2)} - 1_{N_2}) \cdot \mu_2(f) \mu_2(g),
\end{equation}

\begin{equation}
(3.10) \quad \mathbb{E}(\sum_1^1 f(B_j) \sum_1^1 g(C_j)) = 1_{N_{23}} \cdot \mu_2(f) \mu_3(g), \text{ and}
\end{equation}

\begin{equation}
(3.11) \quad \mathbb{E}(f(A_i) \sum_1^1 g(B_j)) = 1_{N_2} \cdot \mu_1(f) \mu_2(g).
\end{equation}

In the expressions to be given below for $1_{N_2}$, $1_{N_2}^{(2)}$, and $1_{N_{23}}$
the symbol $p_{ij}$, $j = 1, 2$, will denote the probability of passing from state 1 to any one of the states 3, 4, 5, ... in one transition and
eventually returning to state \( j \) before the state \( 3 - j \) is reached, \( p_{i \, i \, j} \), \( i, j = 1, 2, 3 \), will denote the probability of passing from state \( i \) to any one of the states \( 4, 5, 6, \ldots \) and eventually returning to state \( j \) before any of the states \( 1, 2, 3 \) other than \( j \) is reached, and \( p_{i \, j \, k} \), \( i, j, k = 1, 2, 3, i \neq j, j \neq k \), will denote the probability of moving from state \( i \) to state \( j \) and then to state \( k \) with possibly intervening transitions to states \( 4, 5, 6, \ldots \) (i.e., \( p_{i \, j \, k} = x(p_{i \, j} + p_{i \, * \, j})(p_{j \, k} + p_{j \, * \, k}) \)).

From elementary algebraic calculations

\[
(3.12) \quad N_2 = \frac{p_{12} + p_{102}}{1 - p_{202}},
\]

\[
(3.13) \quad N_2^{(2)} = \frac{(p_{12} + p_{102})(1 + p_{202})}{(1 - p_{202})^2},
\]

\[
(3.14) \quad N_2^{(3)} = \frac{2p_{232}(p_{121} + p_{131}) + (1 + p_{232})[(p_{12} + p_{1*2})(p_{231} + (p_{13} + p_{1*3})p_{321})]}{(1 - p_{2*2})^2(1 - p_{3*3})^2(1 - p_{232})^3}.
\]

4. Applications.


For the process discussed in the introduction the limiting joint distributions of \( (N_A(t), S_A(t)) \) and \( (N_B(t), S_B(t)) \) can be obtained. \( N_A(t) (N_B(t)) \) is the number of visits to state \( A(B) \) in time \( t \), and \( S_A(t) (S_B(t)) \) is the cumulative occupation time for state \( A(B) \) by time \( t \).

Asymptotically, the vector
(4.1) \[ t^{-1/2}(N_A(t) - \frac{t}{\mu_1 + \mu_2}, S_A(t) - \frac{\mu_1 t}{\mu_1 + \mu_2}) \]

has a bivariate normal distribution with zero mean and covariance matrix \[\sum = (\sigma_{ij})\] where

(4.2) \[ \sigma_{11} = (\sigma_1^2 + \sigma_2^2)/(\mu_1 + \mu_2)^3, \]

(4.3) \[ \sigma_{22} = \left(\mu_2^2 \sigma_1^2 + \mu_1^2 \sigma_2^2\right)/(\mu_1 + \mu_2)^3, \]

(4.4) \[ \sigma_{12} = \sigma_{21} = \left(\mu_1 \sigma_2^2 - \mu_2 \sigma_1^2\right)/(\mu_1 + \mu_2)^3, \]

and the vector

(4.5) \[ t^{-1/2}(N_A(t) - \frac{t}{\mu_1 + \mu_2}, S_B(t) - \frac{\mu_2 t}{\mu_1 + \mu_2}) \]

has a bivariate normal distribution with zero mean, \[\sigma_{11}, \sigma_{22}\] as above, but

(4.6) \[ \sigma_{12} = \sigma_{21} = \left(\mu_2 \sigma_1^2 - \mu_1 \sigma_2^2\right)/(\mu_1 + \mu_2)^3. \]

The limiting distributions of \((N_A(t), N_B(t))\) and \((S_A(t), S_B(t))\) are singular along the lines \((x, x)\) and \((x, -x)\), respectively.

4.2. Zero and One States in an \(M/M/1\) Queue.

Consider a single server queue with Poisson \((\lambda)\) arrivals and negative exponential \((\mu)\) service. If \(X(t)\) is the number of customers in
the queue at time $t$ (including the one in service), then $X(t)$ is a Markov process with countable state space consisting of the non-negative integers. If the zero and one states are singled out for attention and the remaining states 2, 3, 4, ... are considered to be one composite state, then $X(t)$ becomes a semi-Markov process with $m = 3$ states. In the following discussion in order to conform with the notation of the preceding sections the queue zero state will be labelled state 1, the queue state 1 will be labelled state 2, and the composite state consisting of the queue states 2, 3, 4, ... will be labelled state 3.

For this semi-Markov process

$$
\begin{align*}
    p_{12} &= 1, \\
    p_{13} &= 0, \\
    p_{21} &= \frac{\mu}{\lambda + \mu}, \\
    p_{23} &= \frac{\lambda}{\lambda + \mu}, \\
    p_{31} &= 0, \\
    p_{32} &= 1.
\end{align*}
$$

(4.7)

The distributions $F_{ij}$ are independent of $j$ so this semi-Markov process is an example of both sections 3.1 and 3.2. The distribution $F_1$ is negative exponential with parameter $\lambda$, and $F_2$ is negative exponential with parameter $\lambda + \mu$. Provided $\rho = (\lambda/\mu) < 1$ the process is ergodic, and $F_3$ is an honest distribution ($F_3(+\infty)=1$) although it does not have a particularly simple form.

From (3.12) - (3.14) and (4.7),

$$(4.8) \quad \lambda_2 = 1 + \rho, \quad \lambda_3 = \rho(1 + \rho),$$

20
(4.9) \[ _1N_2^{(2)} = (1 + \rho)(1 + 2\rho), \quad _1N_3^{(2)} = \rho(1 + \rho)(2(1 + \rho)^2 - 1), \] and

(4.10) \[ _1N_{23} = 2\rho(1 + \rho)^5/(1 + \rho + \rho^2)^3. \]

If \( f(x) = x \) for \( x \geq 0 \), then

(4.11) \[ \mu_1(f) = 1/\lambda, \quad \mu_2(f) = 1/(\lambda + \mu), \]

(4.12) \[ \mu_3(f) = 1/(\mu - \lambda), \]

(4.13) \[ \sigma_1^2(f) = 1/\lambda^2, \quad \sigma_2^2(f) = 1/(\lambda + \mu)^2, \] and

(4.14) \[ \sigma_3^2(f) = \frac{\lambda + \mu}{(\mu - \lambda)^3}. \]

Let \( N_A(t) \) and \( N_B(t) \) be the number of visits to the queue states 0 and 1, respectively, by time \( t \), and let \( S_A(t) \) and \( S_B(t) \) be the corresponding cumulative occupation times. Pairwise the random variables

(4.15) \[ N_A^*(t) = t^{-1/2}(N_A(t) - \lambda(1 - \rho)t), \]

(4.16) \[ N_B^*(t) = t^{-1/2}(N_B(t) - \lambda(1 - \rho^2)t), \]

(4.17) \[ S_A^*(t) = t^{-1/2}(S_A(t) - (1 - \rho)t), \] and
(4.18) \[ S_B^*(t) = t^{-1/2}(S_B(t) - \rho(1 - \rho)t), \]

have asymptotic bivariate normal distributions with zero mean and variances-covariances given below:

(4.19) \[ \text{Var}(N_A^*(\infty)) = \lambda(1 - 3\rho + 4\rho^2), \]

(4.20) \[ \text{Var}(N_B^*(\infty)) = \lambda(1 + \rho)(1 - \rho - 4\rho^2 + 8\rho^3), \]

(4.21) \[ \text{Var}(S_A^*(\infty)) = 2\rho/\mu, \]

(4.22) \[ \text{Var}(S_B^*(\infty)) = 2\rho(1 - 3\rho + 3\rho^2)/\mu, \]

(4.23) \[ \text{Cov}(N_A^*(\infty), N_B^*(\infty)) = \lambda(1-\rho)^2(1-2\rho^4-2\rho^5-2\rho^6 - 2\rho^2(1+\rho)^3(1-\rho+2\rho^3))/(1+\rho+\rho^2)^3, \]

(4.24) \[ \text{Cov}(S_A^*(\infty), S_B^*(\infty)) = \rho(1-\rho) \left[ - 3 + 6\rho - 5\rho^2 + 3\rho^3 - 2\rho^4 + 2\rho^5 \right. \]

\[ - \frac{2(1+\rho)^4(1 - 3\rho + 4\rho^2 - 2\rho^3)}{(1 + \rho + \rho^2)^3} \] \[ /\mu, \]

(4.25) \[ \text{Cov}(N_A^*(\infty), S_A^*(\infty)) = \rho(3\rho - 1), \]

(4.26) \[ \text{Cov}(N_A^*(\infty), S_B^*(\infty)) = \rho(1 - \rho^2) \left[ 1 - 2\rho + 2\rho^2 - 2\rho^3 - 2\rho^5 \right. \]

\[ - \frac{2\rho^2(1 + \rho)^3(1 - 2\rho + 2\rho^2)}{(1 + \rho + \rho^2)^3} \] \[ \right], \]

22
(4.27) \[ \text{Cov}(N_B^*(\infty), S_A^*(\infty)) = \rho(1 - \rho) \left\{ - 1 + \rho + \rho^2 - \rho^3 + 2\rho^6 \\
- \frac{2\rho(1 + \rho)^4 (1 - 3\rho + 4\rho^2 - 2\rho^3)}{(1 + \rho + \rho^2)^3} \right\}, \]

(4.28) \[ \text{Cov}(N_B^*(\infty), S_B^*(\infty)) = \rho(1 - 2\rho - 2\rho^2 + 7\rho^3). \]

Numerous other applications of the results of sections 2 and 3 can be found. In each example the only complication is the algebraic complexity encountered in computing the asymptotic moments.
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