A CONVERGENT ASYMPTOTIC EXPANSION FOR MILL'S RATIO AND THE NORMAL PROBABILITY INTEGRAL IN TERMS OF RATIONAL FUNCTIONS

BY
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1. Introductory discussion and summary.

In this first of a series of papers, we shall show how certain recent results in asymptotics may profitably be used to develop new expansions for a wide class of integrals, and in particular "tail" probabilities arising in various statistical applications. The expansions are of an asymptotic nature and have the additional advantage that in the statistical applications referred to they are generally also convergent. From a computational point of view this means that the expansions are effective for all values of the argument which are of practical interest. It appears in fact that the expansions are often considerably more accurate than the corresponding "traditional" expansions, when indeed these are available. (Frequently, the expansions developed are of a radically new type so that a basis of comparison is difficult or even impossible.) In the present paper, only the normal probability integral together with the associated Mill's ratio will be considered.

The first known results on the normal integral appear in Laplace's monumental works [11], p. 255 and [12], p. 103. Denote the standardized normal density and distribution functions by $\varphi(\cdot)$ and $\Phi(\cdot)$, respectively, and denote Mill's ratio by $R(\cdot)$, that is, $R(x) = (1 - \Phi(x))/\varphi(x)$. 
Laplace obtains an asymptotic totally divergent series and a convergent rational fraction for $R(x)$, as follows:

\[(1.1) \quad R(x) \sim \frac{1}{x} \left(1 - \frac{1}{x^2} + \frac{1.3}{x^4} - \frac{1.3.5}{x^6} + \ldots \right) \quad (x > 0)\]

and

\[(1.2) \quad R(x) = \frac{1}{x^2} + \frac{1}{x^4} + \frac{2}{x^6} + \frac{3}{x^8} + \ldots \quad (x > 0).\]

(We remark in passing that Laplace actually considered the error function rather than $\Phi(\cdot)$, but (1.1) and (1.2) follow directly from the corresponding Laplacian expansions after a trivial transformation.)

There is an intimate tie-up (cf. Ruben [15]) between (1.1) and (1.2), from the point of view of Stieltjes' classic theory of continued fractions [20], (1.2) being in fact an $S$-summable version of (1.1) through a Stieltjes type continued fraction or $S$-fraction (see Wall [22] for a compact discussion of Stieltjes' summability). Thus a solution to the moment problem$^{1}$

\[(1.3) \quad 1.3.5 \ldots (2j-1) = \int_0^\infty u^j \psi(u) \, du \quad (j = 0, 1, 2, \ldots),\]

where $\psi(\cdot)$ is a bounded non-decreasing function taking on infinitely many values on the half-infinite line $u > 0$ and is constant for $u \leq 0$, is

\[(1.4) \quad \psi(u) = \int_0^u (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}u} t^{-\frac{1}{2}} dt ,\]

---

$^{1}$ The expression 1.3 $\ldots$ (2j-1) is here interpreted as 1 for j=0.
and the series in (1.1) may be expressed in the form

\[ \frac{1}{x} \sum_{j=0}^{\infty} \frac{(-1)^{j}}{2j} \int_{0}^{\infty} u^{j} \, d\psi(u), \]

which formally sums to

\[ (1.5) \frac{1}{x} \int_{0}^{\infty} \frac{d\psi(u)}{1 + u/x^{2}} \quad (x > 0). \]

Consequently, (1.1) may be expressed in terms of the S-fraction (1.2). Moreover, the latter fraction is convergent because of the uniqueness of the moment problem in (1.3) ([22], pp. 362-363), a fact readily proved by Carleman's theorem [6].

According to Stieltjes, integrals of the type (1.5), where \( x \) is real and positive and \( \psi(\cdot) \) is an admissible solution (that is, one having the properties stated after equation (1.3)) of a moment problem such as (1.3) have S-fraction representations with the property that the integrals are greater than every even approximant and less than every odd approximant. In our case, then, (1.2) has the pleasing feature that it yields two convergent sequences which bound \( R(x) \) from below and above, respectively. It is easily shown (for example, by successive integration by parts in \( \int_{x}^{\infty} \psi(t) \, dt \)) and is also well known (e.g., Kendall and Stuart [10], p. 137) that the asymptotic series in (1.1) has a similar enveloping property in the sense that \( R(x) \) is bounded from above by every odd summand and from below by every even summand. However, (1.1) does not give good accuracy except for quite high values of \( x \), whereas (1.2) has been used very effectively by Sheppard [19] in the
derivation of his famous (and posthumously published) tables. In this connection, it is worth noting that Haldane [9] has recently improved on (1.1) by a suitable rearrangement of the terms in the latter expression.

For more recent work on Mill's ratio for the normal integral, reference may be made to Gordon [8], Murty [13], Sampford [17], Tate [21], Boyd [4], Barrow and Cohen [1], Pólya [14], Shenton [18], and Ruben [16]. Gordon obtained \(1/x\) and \(x/(x^2 + 1)\) as upper and lower bounds for \(R(x)\) when \(x > 0\) in the belief that these were not previously available in the literature, while Murty showed how limits to \(R(x)\) of any derived degree of accuracy can be derived for \(x > 0\) by the use of successive convergents of Laplace's continued fraction (1.2).

It is clear that these results are to be regarded as independent rediscovers, since they are merely restatements of the properties of \(S\)-fractions mentioned in the previous paragraph. (In particular, \(1/x\) and \(x/(x^2 + 1)\) are the first two convergents of the fraction (1.2).)

On the other hand, Sampford and Barrow and Cohen obtained some interesting inequalities for various functions of Mill's ratio, while Tate obtained transcendental upper and lower bounds for \(1 - \Phi(x)\), both when \(x \geq 0\) and \(x \leq 0\), and Pólya and Boyd obtained similar upper and lower bounds for \(\Phi(x) - \frac{1}{2}\) and \(R(x)\), respectively, when \(x \geq 0\).\(^2\)

Shenton obtained a continued fraction for \(\int_0^x \frac{\varphi(t)}{\varphi(x)} \, dt\) \((x > 0)\), which

\(^2\) It may be useful to recall here also Birnbaum's conjectured inequality [3] (proved subsequently by Shenton [18]), \(R(x) < 4/(3x+(x^2+8)^{1/2})\) and his earlier inequality [2], \(R(x) > \frac{1}{2}((x^2+4)^{1/2} - x)\), both of these inequalities being valid for \(x \geq 0\). Generalizations of these inequalities have been provided by Shenton [18].
gives more accurate results than the continued fraction (1.2) for small or moderate values of $x$. Finally, Ruben has recently derived a new asymptotic expression for $1 - \Phi(x)$ and the associated $R(x)$ in terms of $\varphi(1/x), \varphi(2/x), \ldots$, when $x > 0$, which is considerably superior to the classical asymptotic expansion (1.1). Ruben's previous expansion is to be contrasted to the present expansion which is (i) not only asymptotic but also convergent for $x > 0$ (as remarked previously) and which (ii) has rational functions for its successive approximants. These approximants provide lower bounds which converge monotonically to $R(x)$, and conservative upper bounds for the difference between any approximant and $R(x)$ are obtained.

2. Derivation of the expansion.

The starting point for the derivation of the required expansion is (1.5). We have shown previously that the latter expression is formally equivalent to the asymptotic expansion (1.1), in the specific sense that (1.5) transforms into (1.1) when $1 + u/x^2$ is expanded as a power series in $u/x^2$ and integration is effected (purely formally and without any attempt at justification) term by term. However, the correspondence between (1.1) and (1.5) is more than formal. It is, in fact, well known (see, for example, Bromwich [5], p. 388, and Kendall and Stuart [10], p. 137) that

\[
R(x) = \frac{1}{\pi} \int_0^\infty \frac{d \psi(u)}{1 + u/x^2} \quad (x > 0),
\]

The integral form for $R(x)$ given by Bromwich and Kendall and Stuart is actually slightly different from that in (2.1), but it reduces to the present integral after a trivial and obvious scale transformation.
where $\psi(\cdot)$ is defined in (1.4). For convenience, set $p = x^2/2$ and apply the transformation

$$
\xi = u/x^2 = u/(2p).
$$

We then obtain

$$
(2\pi)^{\frac{1}{2}} R(x) = \int_{0}^{\infty} e^{-p\xi} \xi^{-\frac{1}{2}} (1+\xi)^{-1} d\xi \quad (x > c).
$$

Term by term integration of the integral in (2.2), after expanding $(1+\xi)^{-1}$ in its power series form, yields the asymptotic expansion (1.1) for $R(x)$. (Watson's lemma [23] ensures that this procedure gives a valid asymptotic expansion.) However, as noted earlier, (1.1) is not very effective except for quite high values of $x$, and we therefore seek for an alternative expansion of the integral in (2.2).

To achieve this objective, consider the more inclusive class of integrals

$$
L_c[f] = L_c[f; p] = \int_{0}^{\infty} e^{-p\xi} \xi^{c-1} f(\xi) d\xi,
$$

where $p$ is real and positive, $c$ may be complex with $\Re(c) > 0$ and $f(\xi)$ is analytic for $\Re(\xi) > 0$. We observe heuristically that if $p$ is large, the behavior of $L_c[f]$ for a given $f$ will be determined overwhelmingly by the behavior of $\xi^{c-1} f(\xi)$ near the origin (this appears plausible because of the rapid diminution of $e^{-p\xi}$ as $\xi$ moves away from the origin). Following Franklin and Friedman [7], a
first approximation to \( L_c[f] \) will be provided by considering only the linear component of \( f(\xi) \) near \( \xi = 0 \). Thus,

\[
L_c[f] = \int_{0}^{\infty} e^{-p^c \xi} \xi^{c-1} (f(0) + \xi f'(0)) \, d\xi
\]

\[
= f(0) \, p^{-c} \Gamma(c) + f'(0) \, p^{-(c+1)} \Gamma(c+1)
\]

\[
= \Gamma(c) \, p^{-c} \left( f(0) + \left( \frac{c}{p} \right) f'(0) \right)
\]

\[
= \Gamma(c) \, p^{-c} \, f(c/p)
\]

Denoting the error of this approximation by \( E_1 \), so that

\[
L_c[f] = \Gamma(c) \, p^{-c} \, f(c/p) + E_1
\]

we have

\[
E_1 = \int_{0}^{\infty} e^{-p^c \xi} \xi^{c-1} (f(\xi) - f(c/p)) \, d\xi
\]

\[
= \int_{0}^{\infty} e^{-p^c \xi} \xi^{c-1} \left( \frac{f(\xi)}{\xi - c/p} \right) (\xi - c/p) \, d\xi
\]

\[
= p^{-1} \int_{0}^{\infty} e^{-p^c \xi} \xi^{c} \frac{d}{d\xi} \left( \frac{f(\xi)}{\xi - c/p} \right) \, d\xi
\]

or integration by parts. The latter integral is of the form (2.3) with \( c \) replaced by \( c+1 \) and \( f(\xi) \) by \( f_1(\xi) = (d/d\xi) \{ (f(\xi) - f(c/p)) / (\xi - c/p) \} \).
Thus,

\[ L_c[f] = \Gamma(c) p^{-c} f(c/p) + p^{-1} L_{c+1}[f], \]

and proceeding in a similar manner we obtain

\[ L_c[f] = p^{-c} \sum_{k=0}^{n-1} \Gamma(c+k) f_k \frac{(c+k-2k)}{p} + p^{-n} L_{c+n}[f], \]

where

\[ f_0(\xi) = f(\xi), \]
\[ f_k(\xi) = \frac{d}{d\xi} f_{k-1}(\xi) - f_{k-1}(\frac{(c+k-1)/p}{\xi}) \quad (k = 1, 2, \ldots). \]

This leads to the following theorem:

**THEOREM I.** (Franklin and Friedman). If \( f(\xi) \) is \( 2n \) times continuously differentiable for \( \xi > 0 \), \( c \) is real and positive, and

\[ |f^{(h)}(\xi)| \leq M e^{\mu \xi} \quad (\xi > 0; h = 0, 1, \ldots, 2n), \]

where \( M \) and \( \mu \) are non-negative constants, then

\[ (2.5) \quad L_c[f] = p^{-c} \sum_{k=0}^{n-1} \Gamma(c+k) p^{-2k} f_k \frac{(c+k)}{p} + O(p^{-2n-c}), \]

as \( p \to \infty \), the \( f_k \) being defined recursively by (2.4).
Reverting to the integral of (2.2), this is seen to be a member of
the class of integrals (2.3) with \( c = \frac{1}{2} \) and \( f(\xi) = (1+\xi)^{-1} \). The condi-
tions of Theorem I are easily verified to hold, so that an asymptotic
expansion, in the specific sense of (2.5), for \( R(x) \) may be deduced.
We shall, however, not obtain the \( F_k \) recursively from (2.4) (a pro-
cedure which is rather tedious) but shall instead obtain them more di-
rectly. This can be achieved by the exploitation of some results used
in the proof of a second theorem of Franklin and Friedman concerning the
convergence of

\[
(2.6) \quad p^{-c} \sum_{k=0}^{\infty} \Gamma(c+k) p^{-2k} f_k(c+k/p)
\]

to \( L_c[f] \) for each \( p > 0 \). As the property of convergence is of in-
trinsic interest and value, we here state the relevant theorem.

**THEOREM II** (Franklin and Friedman). If \( f(z) = f(x+iy) \) can be repre-

dented in the form

\[
(2.7) \quad f(z) = \int_0^\infty e^{-zt} \, d \eta(t) \quad (x > 0),
\]

where \( \eta(t) \) is a complex-valued function which is of bounded variation
in each finite interval \( 0 \leq t \leq T \) and which satisfies the inequality

\[
|\eta(t)| \leq M \quad \text{for} \quad t \geq 0,
\]

then for \( Re(c) > 0 \) the series (2.6) converges to \( L_c[f; p] \) for each
\( p > 0 \).
Reverting once again to (2.2) with \( c = \frac{1}{2} \) and \( f(\xi) = (1+\xi)^{-1} \), observe that

\[
\int_0^\infty e^{-zt} \cdot e^{-t} \, dt = (1+z)^{-1} \quad (\Re(z) > -1),
\]

that is,

\begin{equation}
(2.8) \quad \eta(t) = \int_0^t e^{-u} \, du.
\end{equation}

The further condition in Theorem II on \( \eta(t) \) clearly holds. Thus the series (2.6), with \( c = \frac{1}{2}, \ f_0(u) = (1+u)^{-1} \) and the \( f_k(u) \) for \( k > 1 \) given by (2.4), converges for each \( x > 0 \) to \( (2\pi)^{1/2} R(x) \). (Recall here that \( p = x^2/2 \).)

We shall now obtain (as stated earlier) the \( f_k \) directly. In the proof of Theorem II, Franklin and Friedman show that if \( f(z) \) can be expressed in integral form as given in the latter theorem, then also

\begin{equation}
(2.9) \quad f_k\left(\frac{c+k}{p}\right) = \frac{2k}{k!} \int_0^\infty e^{-ct/p} [g(t/p)]^k \, d\eta(t),
\end{equation}

where

\begin{equation}
(2.10) \quad g(t) = 1 - (1+t)e^{-t}.
\end{equation}
In our case, then, (2.8), (2.9) and (2.10) give

\[ f_k\left(\frac{c+k}{p}\right) = f_k\left(\frac{1+2k}{x}\right) \]

\[ = \frac{x^{2k}}{2^{k}k!} \int_0^\infty e^{-(1+1/x^2)t} \left[ 1 - (1+2t^2/x^2)e^{-2t/x^2} \right] dt \]

(2.11)

\[ = \frac{x^{2k}}{2^{k}k!} \int_0^\infty e^{-(1+1/x^2)t} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (1+x^2t^2/2)^j e^{-2jt/x^2} dt \]

\[ = \frac{x^{2k}}{2^{k}k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \alpha_j, \]

where

(2.12) \[ \alpha_j = \alpha_j(x) = \int_0^\infty e^{-[1+(2j+1)/x^2]t(1+x^2t^2/2)^j} dt. \]

In its turn \( \alpha_j \) can be evaluated after using the binomial expansion of \((1 + x^2t^2/2)^j\) and integrating term by term. This gives after some reduction

(2.13) \[ \alpha_j = \frac{x^2}{x^2+2j+1} \left\{ 1+j \frac{2}{x^2+2j+1} + j(j-1) \left( \frac{2}{x^2+2j+1} \right)^2 + \ldots + j! \left( \frac{2}{x^2+2j+1} \right)^j \right\} \]

\[ (j = 0, 1, \ldots). \]

For convenience, denote the general term in the series (2.6), with
\( c = 1/2, p = x^2/2 \) and the \( f_k \) given through (2.11) and (2.12), by

(2.14) \[ \beta_k = \frac{1 \cdot 3 \cdots (2k-1)}{x^{2k}k!} \cdot \frac{1}{x} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \alpha_j. \]
Then (2.2) shows that

\[(2.15) \quad R(x) = \sum_{k=0}^{\infty} \beta_k(x) \quad (x > 0),\]

where the series (2.15) is both asymptotic and convergent for \(x > 0\) and the \(\beta_k(x)\) are defined in (2.14) and (2.13) as rational functions. This is the main result of the present paper.

3. Evaluation of individual terms of the expansion.

In this section, a convenient recursive method for the evaluation of the \(\beta_k(x)\) will be derived. Unlike the method implied by (2.4), this method does not involve successive differentiation of increasingly complex functions.

Define

\[(3.1) \quad h_k(c) \equiv h_k(c; x) = f_k((c+k)/p) \quad (k = 0, 1, \ldots).\]

From the integral form for \(f_k((c+k)/p)\), where \(c\), though still positive, is here considered arbitrary and \(\eta(t) = e^{-t}\), we find, after integrating by parts followed by some reduction, that

\[(3.2) \quad h_k(c) = \frac{k}{p+c+k} \left\{ h_{k-1}(c) - h_{k-1}(c+1) \right\} \quad (k = 1, 2, \ldots),\]

\[h_0(c) = \frac{p}{p+c}.\]
Defining $\gamma_k(c) = \gamma_k(c; x)$ by

$$
(3.3) \quad \gamma_k(c) = \frac{1}{x} \frac{1 \cdot 3 \cdots (2k-1)}{2^k k!} \eta_k(c) \quad (k = 0, 1, \ldots),
$$

(3.2) gives the following recursion relationship for the $\gamma_k(c)$:

$$
(3.4) \quad \gamma_0(c) = \frac{x}{x^2 + 2c},
$$

$$
\gamma_k(c) = \frac{2k-1}{x^2 + 2c + 2k} \left\{ \gamma_{k-1}(c) - \gamma_{k-1}(c+1) \right\} \quad (k = 1, 2, \ldots).
$$

This last relationship may now be used to evaluate $\gamma_k(c)$ for general $c$, and therefore, in particular, $\beta_k$, on noting that

$$
(3.5) \quad \beta_k = \gamma_k(\frac{1}{2}).
$$

We now evaluate the first three terms in (2.15) with the aid of (3.4) and (3.5).

First,

$$
\beta_0 = \gamma_0(\frac{1}{2}) = \frac{x}{x^2 + 1}.
$$
\begin{align*}
\gamma_1(c) &= \frac{1}{x^2 + 2c + 2} \left\{ \gamma_0(c) - \gamma_0(c+1) \right\} \\
&= \frac{1}{x^2 + 2c + 2} \left\{ \frac{x}{x^2 + 2c} - \frac{x}{x^2 + 2c + 2} \right\} \\
&= \frac{2x}{(x^2 + 2c)(x^2 + 2c + 2)^2}.
\end{align*}

Hence,

\begin{equation}
\beta_1 = \gamma_1^{(1/2)}
\end{equation}

\begin{equation}
= \frac{2x}{(x^2 + 1)(x^2 + 3)^2}.
\end{equation}

To evaluate \( \beta_2 \), we have

\begin{align*}
\gamma_2(c) &= \frac{3}{x^2 + 2c + 4} \left\{ \gamma_1(c) - \gamma_1(c+1) \right\} \\
&= \frac{3}{x^2 + 2c + 4} \left\{ \frac{2x}{(x^2 + 2c)(x^2 + 2c + 2)^2} - \frac{2x}{(x^2 + 2c)(x^2 + 2c + 4)^2} \right\} \\
&= \frac{6x(6x^2 + 12c + 16)}{(x^2 + 2c)(x^2 + 2c + 2)^2(x^2 + 2c + 4)^3}.
\end{align*}

Hence,

\begin{equation}
\beta_2 = \gamma_2^{(1/2)}
\end{equation}

\begin{equation}
= \frac{12x(3x^2 + 11)}{(x^2 + 1)(x^2 + 3)^2(x^2 + 5)^3}.
\end{equation}
From (3.6), (3.7) and (3.8), the first three summands in (2.15) are as follows:

\begin{align*}
(3.9) \quad S_1(x) &= \frac{x}{x^2 + 1}, \\
(3.10) \quad S_2(x) &= \frac{x(x^4 + 6x^2 + 11)}{(x^2 + 1)(x^2 + 3)^2}, \\
(3.11) \quad S_3(x) &= \frac{x(x^{10} + 21x^8 + 176x^6 + 740x^4 + 1611x^2 + 1507)}{(x^2 + 1)(x^2 + 3)^2(x^2 + 5)^3}.
\end{align*}

(Here \( S_n(x) = \sum_{k=0}^{n-1} \beta_k(x) \).)

4. Upper bounds to the truncation errors.

Denote the truncation error induced by the first \( n \) terms of (2.15) as \( E_n(x) \), i.e.,

\begin{equation}
(4.1) \quad E_n(x) = R(x) - \sum_{k=0}^{n-1} \beta_k(x).
\end{equation}

First, since from the integral form for \( f_k((c+k)/p) \) in (2.9) the latter expression is positive, so is \( \beta_k(x) \), and accordingly \( E_n(x) > 0 \). Next, from [7] (eq. (17)),

\begin{equation}
(4.2) \quad E_n(x) = \int_0^\infty S_n(t) \, d\eta(pt),
\end{equation}

where

\begin{equation}
(4.3) \quad S_n(t) = (1+t)^{-\frac{1}{2}} \cdot \frac{\Gamma\left(\frac{1}{2} + n\right)}{\Gamma\left(\frac{1}{2}\right) \Gamma(n)} \int_0^{g(t)} \mu^{n-1}(1-\mu)^{-\frac{1}{2}} \, d\mu
\end{equation}
and \( \eta(t) \) and \( g(t) \) are defined in (2.8) and (2.10), respectively.

Thus

\[
S_n(t) \leq (1+t)^{-\frac{1}{2}} \cdot (1-g(t))^{-\frac{1}{2}} \cdot \frac{\Gamma\left(\frac{1}{2} + n\right)}{\Gamma\left(\frac{1}{2}\right) \Gamma(n)} \int_0^1 g(t) \mu^{n-1} d\mu
\]

\[
= (1+t)^{-\frac{1}{2}} \cdot (1-g(t))^{-\frac{1}{2}} \cdot \frac{\Gamma\left(\frac{1}{2} + n\right)}{\Gamma\left(\frac{1}{2}\right) \Gamma(n)} \frac{(g(t))^n}{n}
\]

\[
= \frac{\Gamma\left(\frac{1}{2} + n\right)}{\Gamma\left(\frac{1}{2}\right) \Gamma(n)} \frac{(1-e^{-t(1+t)})^n}{n} e^{\frac{1}{2}t} (1+t)^{-1}
\]

(4.4)

and the application of (4.4) in (4.2) yields

\[
E_n(x) \leq (2\pi)^{-\frac{1}{2}} \cdot p \cdot \frac{1.3 \ldots (2n-1)}{2^n n!} \int_0^\infty e^{-\frac{1}{2}(p-\frac{1}{2})\xi} (1+\xi)^{-1}(1-e^{-\xi(1+\xi)})^n \, d\xi.
\]

(4.5)

Again,

\[
1 - e^{-\xi(1+\xi)} = \int_0^\xi u \cdot e^{-u} \, du,
\]

whence, since \( u \cdot e^{-u} \) is maximized for \( u > 0 \) at \( u = 1 \),

\[
1 - e^{-\xi(1+\xi)} \leq \xi e^{-1} \quad (\xi \geq 0).
\]

(4.7)

The application of (4.7) in (4.5) yields

\[
E_n(x) \leq (2\pi)^{-\frac{1}{2}} \cdot p \cdot \frac{1.3 \ldots (2n-1)}{2^n n!} \cdot e^{-n} \int_0^\infty e^{-\frac{1}{2}(p-\frac{1}{2})\xi} \xi^n d\xi,
\]

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which reduces to

\[(4.8) \quad 0 < E_n(x) \leq (2\pi)^{-\frac{1}{2}} \cdot 1.3 \cdots (2n-1) \cdot e^{-n} \cdot \frac{x^2}{(x^2-1)^{n+1}},\]

provided \( p > \frac{1}{2} \), i.e., \( x > 1 \).

An alternative upper bound to \( E_n(x) \) is provided by the inequality (refer to (4.6))

\[(4.9) \quad 1 - e^{-\xi (1+\xi)} \leq \int_0^\xi u \, du = \frac{1}{2} \xi^2 \quad (\xi \geq 0),\]

which gives with the aid of (4.2) and (4.4),

\[(4.10) \quad 0 < E_n(x) \leq (2\pi)^{-\frac{1}{2}} \cdot \frac{1}{n!} [1.3 \cdots (2n-1)](2n)! \frac{x^2}{(x^2-1)^{2n+1}},\]

provided \( x > 1 \).

We remark that neither of the two upper bounds in (4.8) and (4.10) is uniformly better than the other in the sense of not exceeding it for all \( x > 0, \ n \) being fixed. In fact the ratio of the upper bound in (4.8) to that in (4.10) is

\[(4.11) \quad \frac{e^{-n}}{(n+1)(n+2) \cdots (2n)} \frac{x^2}{(x^2-1)^n},\]

which is \(< 1\) or \(> 1\) according as to whether \( x^2-1 \) is, or is not less than \( [(n+1)(n+2) \cdots (2n)]^{1/n} \). In other words, (4.8) should be used for a given number of terms \( n \), in preference to (4.10), if \( x^2-1 \) is less than the latter quantity, whereas (4.10) is preferable if the
reverse is the case, i.e., for sufficiently large $x$. In practice, both
bounds should be computed for given $n$ and $x$, and the final upper
bound taken as the minimum of the two upper bounds. However, it appears
that the latter bounds are conservative and it would be desirable to ob-
tain sharper bounds for (4.2).

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