SEQUENTIAL TEST FOR THE MEAN OF A NORMAL DISTRIBUTION III (small t)

BY

HERMAN CHERNOFF

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In honor of my teacher, colleague, and friend

Professor Charles Loewner

on his 70th birthday

1. **Summary.**

Asymptotic expansions are derived for the behavior of the optimal sequential test of whether the unknown drift $\mu$ of a Wiener-Levy process is positive or negative for the case where the process has been observed for a short time. The test is optimal in the sense that it is the Bayes test for the problem where we have an a priori normal distribution of $\mu$, the regret for coming to the wrong conclusion is proportional to $|\mu|$ and the cost of observation is one per unit time.

The Bayes procedure is compared with the best sequential likelihood ratio test and with the procedure which calls for stopping when no fixed additional sampling time is better than stopping. The derivations allow for generalizing to variations of this problem with different cost structure.

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1/ Some of this work was carried out while the author was on leave at the London School of Economics and Political Science and at the University of Rome with the partial support of ONR contract Nonr 3439(00), NR 342-025.
2. **Introduction.**

In the Fourth Berkeley Symposium on Probability and Statistics Chernoff [3] indicated that the problem of sequentially testing whether the mean drift of a Wiener-Levy process is positive or negative, given a normal *a priori* probability distribution, was relevant to the problem of deriving an asymptotically (as sampling cost approaches zero) optimal sequential test of whether the mean of a normally distributed variable is positive or negative. The former problem was reduced to the solution of a free boundary problem involving a diffusion equation.

Subsequently Moriguti and Robbins [7] and Bather [1] derived asymptotic expansions for the optimal procedure and Bayes risk for time $t$ large, i.e., for the case where the process has been observed for a long time. Breakwell and Chernoff [2] also derived these expansions and proved that they were valid in the sense of yielding asymptotic approximations to the desired optimum boundary and the corresponding risk. In section 3 of the latter paper there is a brief review of the results and notation of [3], which is urgently recommended to the reader of the present paper. The case where $t \to 0$ is studied. Although this corresponds to the beginning of sampling in the normalized problem, it is relevant to large sample situations when the cost of sampling is small. In fact, suppose that in a realistic version of the problem the scales are such that each observation has variance 1, the cost of making the wrong decision is roughly $10^3|\mu|$, the *a priori* probability of $\mu$ has variance 1 and the cost per observation is $10^{-3}$. Then, in the normalized problem the starting point is set at $(\mu_0, t_0)$ where $t_0 = 10^{-4}$ and the discrete problem involves time steps of length
\( \delta = 10^{-4} \). Then after a substantial number of observations one will still be in a case of \( t \) small.

In section 3, a formal derivation will be presented of expansions for the free boundary and the Bayes risk for the optimal solution. This derivation involves a special solution of the heat equation. The motivation for the use of this solution is briefly indicated in subsequent sections but can be bypassed and is no longer required. In section 4 alternative expansions are presented. These have the advantage of being easier to compute and somewhat more elegant.

The problem of proving that these expansions do represent asymptotic approximations to the solution for small \( t \) is then attacked. In section 5 it is shown that the expansion for the boundary is an asymptotic approximation. In section 6 the same is done for the Bayes risk. The latter proof involves a representation of solutions of the heat equation which appears in Goursat [6]. This representation is particularly well adapted to our free boundary problem for it involves the value of the solution and its partial derivatives on the boundary both of which are described by our boundary conditions. The proofs in these two sections have been complicated substantially by the requirement that they be adaptable to variations of the stated problem. Simpler proofs could be presented which apply the symmetry of the special stated problem.

Section 7 is devoted to methods of deriving bounds on the behavior of the boundary and risk. This section serves two purposes. First it indicates some of the motivation which led to the formal expansions and the arguments used in preceding sections. Second, we describe the
boundary and risk for the procedure which consists of stopping at a
point where there is no fixed sampling time which will do better than
stopping. This procedure arises naturally in this section because it
leads to stopping early, thereby providing useful bounds. At the same
time it is relatively easy to evaluate.

In section 8 we compare the optimal procedure with the best pro-
cedure of the Wald type when the a priori probability distribution of
μ has large variance. In section 9 we decompose the Bayes risk of
the optimal procedure into a Sampling risk and risk due to Error.
Finally in section 10, we discuss briefly the important question of the
operating characteristics of the optimal procedure. That is to say we
comment on the expected sample size and error probability corresponding
to a specified μ when the Bayes procedure is applied. These character-
istics are fundamental to a proper evaluation of the procedure for cases
where the a priori normal assumption is not necessarily valid.

To facilitate reference we terminate this introduction by supple-
menting the reference to [2, §3] by a brief outline of some of the
notation we shall use. The statistician observes \( X_t = \mu t + Z_t \)
where \( Z_t \) is a Wiener-Levy process of independent Gaussian increments with
mean zero and variance one per unit time. The a priori probability
distribution of \( \mu \) and a starting point \((x_0, t_0)\) are chosen so that
given \( X_t = x \), the a posteriori probability distribution of \( \mu \) is
normal with mean \( x t^{-1} \) and variance \( t^{-1} \), i.e., \( \mathcal{N}(x t^{-1}, t^{-1}) \). Then
we represent the risk due to additional sampling and error probability
by \( B(x, t) \) and the risk due to stopping by \( D(x, t) \). In our problem
\[ D(x,t) = t^{-1/2} \psi(xt^{-1/2}) \text{ where } \psi(\alpha) \text{ is symmetric and } \]

\[ \psi(\alpha) = \varphi(\alpha) - \alpha[1-\Phi(\alpha)] \text{ for } \alpha > 0. \]

For a given procedure represented by a continuation region, \( B \) satisfies the equation \( 1 + B_t + xt^{-1}B_x + \frac{1}{2} B_{xx} = 0 \) subject to the restriction \( B = D \) on the boundary. For the optimal procedure \( B = D \) on the boundary. We shall find it convenient to transform to \( y = xt^{-1} \) and \( s = t^{-1} \). In these variables the differential equation and boundary conditions for \( u(y,s) = t + B(x,t) \) are

\[ u_t = \frac{1}{2} u_{yy} \]

\[ u = s^{-1} + s^{1/2} \varphi(ys^{-1/2}) \text{ on the boundary} \]

\[ u_y = -[1-\Phi(ys^{-1/2})] \text{ " " " } \]

The variable \( \alpha = ys^{-1/2} = xt^{-1/2} \) will be used throughout the paper.
3. A formal expansion.

To derive our formal expansion we shall consider solutions of the heat equation of a special form and apply the two boundary conditions alternately to obtain successive terms in the expansions for the boundary and optimal risk. The proof of the validity of these expansions is given in sections 5 and 6, and some remarks on the motivation appear in those sections and section 7. The special form of solution of the heat equation is

\( u = Ks^{-1/2} \varphi(\alpha) + 2\varphi(\alpha) \int_0^\infty e^{-b^2/2} \left[ \cosh \alpha b - 1 \right] f(s^{1/2} b) db \) \hfill (3.1)

while the boundary will be represented by an expansion of the form

\( \beta = \log s = \frac{\alpha^2}{2} + k_0 + k_1 \alpha^{-2} + k_2 \alpha^{-4} + \ldots = \frac{\alpha^2}{2} + k_0 + \eta \) \hfill (3.2)

First we indicate that \( u \) is a solution of the heat equation. It is easy to see that \( s^{-1/2} \varphi(s^{-1/2} y) \) is a solution for \( s > 0 \). Similarly \( \int s^{-1/2} \varphi(s^{-1/2} (y-y')) dF(y') \), which is a heat potential with sources distributed along the \( y \) axis is a solution provided the integral converges. A mass of \( K \) for \( dF \) at \( y' = 0 \) provides the \( Ks^{-1/2} \varphi(\alpha) \) term of the proposed solution. Elsewhere let \( dF/dy' \) be represented by the symmetric density \( f(y') \) which is compensated for by the mass \( -f(y') dy' \) at \( y' = 0 \). Setting \( y' = s^{1/2} b \), this process yields

\( g(\alpha, s) = \int_0^\infty \left[ \varphi(\alpha-b) + \varphi(\alpha+b) - 2\varphi(\alpha) \right] f(s^{1/2} b) db \) \hfill (3.3)

\[ = 2 \varphi(\alpha) \int_0^\infty e^{-b^2/2} \left[ \cosh \alpha b - 1 \right] f(s^{1/2} b) db \]
To assure convergence to a solution some regularity conditions are required. For the above representation it suffices that (i) \( f \) be measurable while \( f(y)\exp(-cy^2) \to 0 \) as \( y \to \infty \) for each \( c > 0 \) and (ii) \( y^2f(y) \) be absolutely integrable in every finite interval. Thus \( f(y) \) may approach infinity as \( y \to 0 \) like \( y^{-2}\log y \) without causing difficulty.

Now we note that along the boundary, the term \( K s^{-1/2} \varphi(\alpha) \) is relatively negligible. To be more specific, along the boundary represented by (3.2), \( \alpha = s^{-1/2} y = [3 \log s + 0(1)]^{-1/2} \). Then \( s^{-1/2} \varphi(\alpha) \sim s^{-2} \) and its partial derivative with respect to \( y \), \( -s^{-1} \varphi(\alpha) \sim s^{-5/2} (\log s)^{1/2} \). On the other hand \( s^{-1} + s^{1/2} \psi(\alpha) \sim s^{-1} \) and \( 1 - \psi(\alpha) \sim s^{-3/2} (\log s)^{-1} \) along the boundary. From this it will follow that at the boundary the main part of \( u \) and \( u_y \) is contributed by \( g \).

The function \( g \) may be approximated for large \( \alpha \) by use of a simple argument similar to that used in the method of steepest descent. For large \( \alpha \), we have the formal expansions

\[
(3.4a) \quad g(\alpha, s) \approx E^*[f(s^{1/2}(\alpha+\varepsilon))] = f(s^{1/2} \alpha) + \frac{s}{2!} f^{(2)}(s^{1/2} \alpha) + \frac{1 \cdot 3}{4!} s^2 f^{(4)}(s^{1/2} \alpha) + \frac{1 \cdot 3 \cdot 5}{6!} s^3 f^{(6)}(s^{1/2} \alpha) + \ldots
\]

and

\[
(3.4b) \quad g(\alpha, s) \approx E^*[cf(s^{1/2}(\alpha+\varepsilon))] = s^{1/2} f(1)(s^{1/2} \alpha) + \frac{1 \cdot 3}{3!} s^2 f^{(3)}(s^{1/2} \alpha) + \frac{1 \cdot 3 \cdot 5 \cdot 7}{5!} s^3 f^{(5)}(s^{1/2} \alpha) + \ldots
\]

where \( \varepsilon \) is regarded as normally distributed with mean 0 and variance 1 and \( E^* \) is the expectation operator applied after the operand has been expanded in a power series in \( \varepsilon \). These expansions are asymptotic approximations if the derivatives of \( f \) are sufficiently well behaved.
Neglecting the $Ks^{-1/2}\phi(\alpha)$ term, the boundary conditions of (2.1) transform to

\begin{align}
(3.5a) \quad E*[s^{1/2}(\alpha+\epsilon)] &= s^{-1} + s^{1/2}\psi(\alpha) \\
(3.5b) \quad E*[e^2[s^{1/2}(\alpha+\epsilon)] &= -s^{1/2}[1-\phi(\alpha)] .
\end{align}

Substituting (3.2) in the right hand side of (3.5) and using the expansions [5, p. 179]

\begin{align}
(3.6a) \quad 1-\phi(\alpha) &= \phi(\alpha)[\alpha^{-1} - \alpha^{-3} + (1.3)\alpha^{-5} - 1.35\alpha^{-7} + \cdots ] \\
(3.6b) \quad \psi(\alpha) &= \phi(\alpha)[\alpha^{-2} - 1.3\alpha^{-4} + 1.35\alpha^{-6} \cdots ] ,
\end{align}

we have the boundary conditions

\begin{align}
(3.7a) \quad E*[f[s^{1/2}(\alpha+\epsilon)]] &= s^{-1}[1+(2\pi)^{-1/2}e^{3k/2}e^{3\theta/2}[\alpha^{-2}-3\alpha^{-4}+15\alpha^{-6} \cdots ]] \\
(3.7b) \quad E*[e^2f[s^{1/2}(\alpha+\epsilon)]] &= -s^{-1}(2\pi)^{-1/2}e^{3k/2}e^{3\theta/2}[\alpha^{-1}-\alpha^{-3}+3\alpha^{-5}-15\alpha^{-7} \cdots ].
\end{align}

Let us compute a few terms in the apparently natural expansions of $f$ and $\beta$ which satisfy (3.7) in order to see how the other terms are derivable. Starting with the approximation

\begin{equation}
(3.8a) \quad f_0(x) = 6x^{-2}\log x = 3x^{-2}\log x^2 ,
\end{equation}

the $s^{-1}$ terms match in equation (3.7a) and (3.7b) leads to

\begin{equation}
(2\pi)^{-1/2}\exp(3k/2) = 2 \quad \text{or}
\end{equation}

\begin{equation}
(3.8b) \quad k_0 = (\log 8\pi)/3 .
\end{equation}
We apply the resulting approximation

\begin{equation}
\beta \approx \beta_0 = \frac{\alpha^2}{2} + \frac{\log 8\pi}{3}
\end{equation}

(3.9)

to (3.7a), substituting $2\alpha^2$ for $x^2$ in the argument of $f$ and we obtain a discrepancy [right side of (3.7a) minus left side] which is

\begin{equation}
-(s\alpha^2)^{-1}\left(3 \log \alpha^2 + \log 8\pi + 1 + O*(\alpha^2)\right)
\end{equation}

(3.10)

where $O*(\alpha^{-2r})$ is used to represent an expression which is bounded by some power of $\log \alpha^2$ divided by $\alpha^{2r}$ as $\alpha \to \infty$. To compensate for this discrepancy we apply a correction to $f_0$ making use of the approximations $s\alpha^2 \approx x^2$, $\log \alpha^2 \approx \log[3 \log x^2]$. This gives

\begin{equation}
f_1(x) = x^{-2}\left(3 \log x^2 - 3 \log[\log x^2] - 3 \log 3 - \log 8\pi - 1\right).
\end{equation}

(3.11)

This approximation combined with (3.9) yields a discrepancy in (3.7b) which is

\begin{equation}
2(s\alpha)^{-1}\left(3\alpha^{-2} + O*(\alpha^{-4})\right)
\end{equation}

(3.12)

and the main part of this can be compensated for by the approximation

\begin{equation}
\beta = \frac{\alpha^2}{2} + \frac{\log 8\pi}{3} + 2\alpha^{-2}.
\end{equation}

(3.13)

We are now in a position to state and prove the following theorem about the existence of a formal expansion.
Theorem 3.1. There are expansions of the form

\[ (3.14a) \quad \beta = \log s \sim \frac{\alpha^2}{3} + \frac{\log 8\pi}{3} + K_1 \alpha^{-2} + K_2 \alpha^{-4} + \cdots = \frac{\alpha^2}{3} + \frac{\log 8\pi}{3} + \eta \]

and

\[ (3.14b) \quad f(x) \sim 3x^{-2}(\log x^2)(1 + P_1 (\log x^2)^{-1} + P_2 (\log x^2)^{-2} + \cdots) \]

where the \( K_r \) are polynomials in \( \log \alpha^2 \) and the \( P_r \) are polynomials in \( \log[\log x^2] \) with the following property. If \( \beta_r \) and \( f_r \) represent the above expressions with the sums terminated with \( K_r \alpha^{-2r} \) and \( P_r (\log x^2)^{-r} \), substitution in equations (3.5a) and (3.5b) leave discrepancies of \( s^{-1}0(\alpha^{-2-2r}) \) and \( s^{-1}0(\alpha^{-3-2r}) \) respectively.

Proof: First we note that substituting \( \beta_r \) and \( f_r \) in both sides of (3.7a) and (3.7b) leads to asymptotic expansions. Each side of (3.7a) can be represented by an expansion of the form

\[ s^{-1}(1 + Q_1 \alpha^{-2} + Q_2 \alpha^{-4} + \cdots) \]

where \( Q_1 \) is a generic polynomial in \( \log \alpha^2 \). Each side of (3.7b) can be represented by an expansion of the form

\[ -2(s\alpha)^{-1}(1 + Q_1 \alpha^{-2} + Q_2 \alpha^{-4} + \cdots) . \]

Of course the generic polynomials \( Q_1 \) are not to be assumed to be equal to one another.

Let us assume that the theorem holds for a certain value of \( r \geq 0 \). As we have already seen, it does hold for \( r = 0 \) and \( 1 \). We observe that adding \( x^{-2}(3 \log x^2)^{-r}P(\log(3 \log x^2)) \) to \( f_r \) has the
effect of increasing the left side of (3.7a) by \( s^{-1} \alpha^{-2-2r} p(\log \alpha^2) [1 + O*(\alpha^{-2})] \). Furthermore it has the effect of changing the left side of (3.7b) by \( s^{-1} O*(\alpha^{-3-2r}) \). It does not affect the right hand sides of these equations.

Since the theorem applies for \( r \), the first \( r \) \( Q \)'s match on both sides for the expansions of (3.7a) and (3.7b). By letting \( P \) be the discrepancy of the two \( Q_{r+1} \) for equation (3.7a) the resulting \( f \) which we call \( f_{r+1} \) reduces the discrepancy in that equation to \( s^{-1} O*(\alpha^{-4-2r}) \). Equation (3.7b) still has a discrepancy of \( s^{-1} O*(\alpha^{-3-2r}) \).

Now we observe that adding \( K(\log \alpha^2) \alpha^{-2r-2} \) to \( \beta_r \) does not change the first \((r+1)\) \( Q \)'s of the left hand sides of equations (3.7). Basically this is so because the change in \( \beta \) is equal to \( \beta_r O*(\alpha^{-4-2r}) \), the change in \( \log s \alpha^2 \) is equal to the change in \( \beta \), and the change in \( \log[\log s \alpha^2] \) is \( O*(\alpha^{-4-2r}) \). On the other hand the right hand sides are changed by

\[ 3s^{-1} \alpha^{-4-2r} K(\log \alpha^2) [1 + O*(\alpha^{-2})] \text{ and } -3s^{-1} \alpha^{-3-2r} K(\log \alpha^2) [1 + O*(\alpha^{-2})] \] respectively. Thus by setting \( K \) equal to the difference of the \( Q_{r+1} \) on the two sides of equation (3.7b) we can match the first \((r+1)\) \( Q \)'s of both sides of this equation without affecting the corresponding \( Q \)'s of (3.7a). Hence the result holds for \( r+1 \), and the theorem follows.

It should be noted that \( f_r \) is not defined for all \( x \). If we replace \( x^2 \) by \( x^2+2 \) in the expression (3.14b) then \( \log[\log x^2] \) and hence the new \( f_r \) are defined for all \( x \). All of the asymptotic expansions of this section are unaffected. Furthermore the integral (3.3) is now defined when \( f \) is replaced by the new \( f_r \).

In section 5 we shall study the corresponding \( g \). We remark here merely that while \( g \) is large compared to \( Ks^{-1/2} \varphi(\alpha) \) near the boundary, it is relatively small, \( O(s^{-1} \log s) \), for bounded \( \alpha \).
4. An alternative formal expansion.

An alternative method of obtaining a formal expansion is somewhat easier for computational purposes. Essential use is made of the function \( f_0^*(x) \) which is defined so that

\[
f_0^*(x) = s_1^{-1}
\]

when

\[
x = [s_1(3 \log s_1 - \log 8\pi)]^{1/2}.
\]

This function was selected because \( s_1^{-1} + s_1^{1/2} \Psi(\alpha) = s_1^{-1} \) along the boundary and, using the approximation \( \beta_0 \) of the previous section, i.e., \( \log s \approx \frac{\alpha^2}{3} + \frac{1}{3} \log 8\pi \), we have

\[
s_1^{1/2} \alpha \approx s_1^{1/2} [3 \log s - \log 8\pi]^{1/2}
\]

and hence

\[
\mathbb{E}*[f_0^*[s_1^{1/2}(\alpha+\epsilon)]] \approx f_0^*[s_1^{1/2} \alpha] \approx s_1^{-1}.
\]

A tedious computation indicates that an analogue of Theorem 3.1 applies using expansions of the form

\[
(4.2a) \quad f(x) = f_0^*(x) + c_1 x^{-2} + c_2 x^{-4} [f_0^*(x)]^{-1} + c_3 x^{-6} [f_0^*(x)]^{-2} + \ldots
\]

and

\[
(4.2b) \quad \alpha^2 = 3\nu[1 + k_1 w^{-2} + k_2 w^{-3} + \ldots]
\]

where the \( c_1 \) and \( k_1 \) are constants and
\[(4.2c) \quad w = \log s - \frac{1}{3} \log 8\pi\]
corresponds to the approximation for \(a^{2/3}\) based on \(\beta_0\). The constants thus far evaluated are

\[(4.3a) \quad c_1 = -1, \ c_2 = -3, \ c_3 = \frac{21}{2}, \ c_4 = \frac{-3(193)}{2}, \ c_5 = \frac{9(179)}{4}, \ c_6 = \frac{-27(28,937)}{20}\]

and

\[(4.3b) \quad k_1 = \frac{-2}{3}, \ k_2 = \frac{-5}{9}, \ k_3 = \frac{-22}{9}, \ k_4 = \frac{-1627}{54}, \ k_5 = \frac{-16184}{405}, \ k_6 = \frac{-17152}{81}\]

The following consideration is basic to the convenient computation of the above constants. If

\[z = -\log f_0^*(x) - \frac{1}{3} \log 8\pi,\]
\[z + \log z = \log \frac{x^2}{2} - \frac{1}{3} \log 8\pi,\]
\[\frac{dz}{dx} = (1+z^{-1})^{-1} \frac{2}{x},\]
and the derivative of \([f_0^*(x)]^{-\frac{r_2}{2}}\) \(z^{-\frac{r_3}{3}}\) is \([f_0^*(x)]^{-\frac{r_1}{x}} - (r_2+1)\)
multiplied by a series in descending powers of \(z\).
5. Asymptotic approximation to the optimal boundary.

The object of this section is to establish that the formal expansions of sections 3 and 4 yield asymptotic approximations to the optimal boundary. The proof will be preceded by several lemmas that detail properties of the function $g$ defined by

\begin{equation}
(5.1) \quad g(\alpha, s) = \int_0^\infty [\varphi(\alpha-b) + \varphi(\alpha+b) - 2\varphi(\alpha)]f(s^{1/2}b)db
\end{equation}

where

\begin{equation}
(5.2) \quad f(x) = 3(x^2+2)^{-1/2}\log(x^2+2)[1+\log(x^2+2)]^{-1}p_1 + \cdots + [\log(x^2+2)]^{-r}p_r
\end{equation}

and the $p_i$ are polynomials in $\log[\log(x^2+2)]$. Several related quantities which are studied are

\begin{equation}
(5.3) \quad \Delta_K(\alpha, s) = g(\alpha, s) + K^{1/2} \varphi(\alpha) - s^{-1/2} \psi(\alpha),
\end{equation}

and $\hat{\alpha}_K(s)$, the positive value of $\alpha$, if any, for which $\Delta_K(\alpha, s)$ is maximized. Let $\theta_K(s)$ be the maximum value. Finally let $\alpha^*_K(s)$ be the greatest value of $\alpha$, if any, for which $0 \leq \alpha^*_K(s) \leq \hat{\alpha}_K(s)$ and $\Delta_K(\alpha^*_K(s), s) = 0$.

**Lemma 5.1** (a) As $s \to \infty$,

\begin{equation}
(5.4) \quad \sup_{\Delta} \left| \frac{\partial^i g(\alpha, s)}{\partial \alpha^i} \right| = o(s^{-1/2} \log s) \quad \text{for each integer } i=0,1,2,\ldots.
\end{equation}

(b) For fixed $s$ and large $\alpha$, and each integer $i=0,1,2,\ldots$

\begin{equation}
(5.5) \quad \frac{\partial^i g(\alpha, s)}{\partial \alpha^i} \approx (-1)^i 3(i+1)!(s\alpha^2)^{-1} \log(s\alpha^2).
\end{equation}
(c) Equation (5.5) also holds for large \( s \) when \( \alpha^2/\log s \) is bounded away from zero.

Proof: First we establish (b). Fix \( s \) and let \( \alpha \to \infty \). Then

\[
\frac{\partial^i g}{\partial \alpha^i} = \int_0^\infty \frac{\partial^i f(\alpha-b)}{\partial \alpha^i} f(s^{1/2}_b) db + O(\alpha)
\]

\[
= F^*(H^*_1(\epsilon)f(s^{1/2}(\alpha+\epsilon))]
\]

where \( H^*_i(\epsilon) \) is the \( i \)th order Hermite Polynomial in \( \epsilon \), which is orthogonal to (uncorrelated with) the first \( i \) powers of \( \epsilon \). Hence

\[
\frac{\partial^i g}{\partial \alpha^i} \approx s^{1/2} \frac{\partial^i f}{\partial \alpha^i} [s^{1/2}\alpha] \approx (-1)^i 3! (i+1)! (s\alpha^{2i+1})^{-1} \log(s\alpha^2).
\]

Now we proceed to (a). For convenience we take \( s > e \).

\[
(5.6) \quad \frac{\partial^i g}{\partial \alpha^i} = \frac{1}{s} \int_0^\infty \frac{1}{b^2} \frac{\partial^i}{\partial \alpha^i} [f(\alpha-b)+f(\alpha+b)-2f(\alpha)] \cdot sb^2 f(s^{1/2}_b) db.
\]

For \( 0 \leq b \leq 1, |b^{-2} \frac{\partial^i [f(\alpha-b)+f(\alpha+b)-2f(\alpha)]}{\partial \alpha^i}| \leq J_1(\alpha) \exp[\alpha - \frac{1}{2} \alpha^2]
\]

where \( J_1(\alpha) \) is a polynomial in \( \alpha^2 \). Also \( |sb^2 f(s^{1/2}_b)| = J_2 \log s \)

where \( J_2 \) is a constant. Hence the interval from \( 0 \) to \( 1 \) contributes to (5.6) an amount bounded by \( J_1(\alpha) J_2 \exp(\alpha - \frac{1}{2} \alpha^2) s^{-1} \log s = O(s^{-1} \log s) \).

For \( b > 1, |f(s^{1/2}_b)| \leq J_3 (sb^2)^{-1} \log sb^2 \leq J_3 s^{-1} \log s \) where \( J_3 \) is a constant. The partial derivatives in the integrand are uniformly bounded and (a) follows.

For the sake of brevity we shall prove (c) only for the case \( i=0 \).

Take \( s > e, \alpha > 2 \) and for given \( \xi, 0 < \xi < 1, \) we decompose the integral into the parts from \( 0 \) to \( 1 \) to \( \alpha(1-\xi) \) to \( \alpha(1+\xi) \) to \( \alpha. \)
The contribution from 0 to 1 is bounded as above by
\[ J_1(\alpha)J_2\exp\left(\frac{b^2}{2}\right)\frac{1}{s^2} \log s. \] The contribution from 1 to \(\alpha(1+)\) is bounded by \(J_1s^{-1}\log s \Phi(-\alpha^2)\) where \(J_1\) is a constant. The same bound applies to the part from \(\alpha(1+)\) to \(\infty\). Finally for \(\alpha(1+) < b < \alpha(1+)\),

\[ \left| f(s^{1/2}b)/f(s^{1/2}\alpha) - 1 \right| \leq J_2(\xi) \] for \(s > s(\xi)\) and \(\alpha > \alpha(\xi)\) where \(J_2(\xi) \to 0\) as \(\xi \to 0\). It follows that given \(\xi > 0\), there exists \(\xi(\xi)\) such that for \(s > s(\xi)\) and \(\alpha > \alpha(\xi)\),

\[ \left| \int_{\alpha(1-\xi)}^{\alpha(1+) \left[ \phi(\alpha-b)+\phi(\alpha+b)-2\phi(\alpha) \right] f(s^{1/2}b)db \right| \leq \xi f(s^{1/2}\alpha). \]

Combining the various segments, we have

\[ |g(\alpha,s) - f(s^{1/2}\alpha)| \leq \xi f(s^{1/2}\alpha) + J_3(\alpha)s^{-1}\log s \exp(-\alpha^2 \xi^2/2), \]

which implies \(c\) for \(i=0\). For \(i > 0\), \(f\) must be treated more carefully for the section \(\alpha(1-\xi)\) to \(\alpha(1+)\).

**Lemma 5.2.** (a) For large \(s, \frac{1}{\alpha}\)

\[ (5.7) \quad \hat{\alpha}_K = \left\{ 3 \log s - \log 8\pi + o(1) \right\}^{1/2}, \]

\[ \frac{\partial^2 \Delta_K}{\partial \alpha^2} < 0 \quad \text{and} \quad \frac{\partial \Delta_K}{\partial \alpha} > 0 \text{ for } \alpha < \hat{\alpha}_K \text{ while } \frac{\partial \Delta_K}{\partial \alpha} < 0 \text{ for } \alpha > \hat{\alpha}_K. \]

(b) For \(\alpha^2 = 3 \log s - \log 8\pi + C + o(1)\),

\[ \frac{\partial^2 \Delta_K}{\partial \alpha^2} \approx -2s^{-1}e^{-c/2}. \]

(c) Decreasing \(K\) increases \(\hat{\alpha}_K\) and decreases \(\theta_K(s)\) monotonically and continuously.

---

\(^1/\) In this lemma "large \(s\)" may depend on \(K\) as well as the particular form of \(f\).
(d) For large \( s \), \( \alpha_k^* \approx \sqrt{3} \log s \) exists if \( \theta_k(s) > 0 \) and is nonexistent if \( \theta_k(s) < 0 \). In the former case, increasing \( K \) decreases \( \alpha_k^*(s) \) monotonically.

(e) For large \( s \), \( \alpha_k^* < 0 \) for \( 0 < \alpha < \alpha_k^*(s) \) when \( \alpha_k^*(s) \) exists positive.

Proof: The first and second derivatives of \( Ks^{-1/2} \varphi(\alpha) \) with respect to \( \alpha \) are \( -Ks^{-1/2} \alpha \varphi(\alpha) \) and \( Ks^{-1/2}(\alpha^2-1) \varphi(\alpha) \) respectively. The first and second derivatives of \( s^{-1/2} (s^2 s^{-1/2} \psi(\alpha)) \) with respect to \( \alpha \) are \( -s^{1/2} [1 - \varphi(\alpha)] = -s^{1/2} \alpha^{-1} \varphi(\alpha) \) and \( s^{1/2} \varphi(\alpha) \) respectively. Applying lemma (5.1a) for \( \alpha^2 < \log s \) and lemma (5.1c) for \( \alpha^2 > \log s \). Parts (a) and (b) follow, parts (c), (d), and (e) are trivial.

Consider the function \( f \) where the \( P_i \) are obtained from the expansion of Theorem 5.1, with \( x^2 \) replaced by \( x^{2+2} \). Adding (subtracting) the constant one to (from) \( P_r \), we designate the resulting function \( f^+(f^-) \). We label the corresponding \( g, \Delta, \beta, \alpha^*, \theta \) etc. with the superscript \( +(-) \) and the subscripts \( Kr \). Let \( \beta_r^+ \) represent the sum of the terms thru \( K_r \alpha^{-2r} \) in the expansion (5.14a) and let \( \alpha_r^+(s) \) represent the inversion of \( \beta_r^+ \). Our object is to prove that the optimal boundary \( \tilde{\alpha} \) is approximated by \( \alpha_r^+(s) \).

Lemma 5.3 For \( s \) large,
\[
\theta_{Kr}^+(s) \approx s^{-1}(\log s)^{-r}, \quad \theta_{Kr}^-(s) \approx -s^{-1}(\log s)^{-r}
\]
\[
\varphi_{Kr}^+(s) = \alpha_r^+o(\log s)^{-r}, \quad \varphi_{Kr}^-(s) = \alpha_r^-o(\log s)^{-r}
\]
\[
\alpha_{Kr}^+(s) \approx \alpha_r^-(\log s)^{-r/2}
\]
Furthermore both \( \theta_{Kr}^+(s) \) and \( \theta_{Kr}^-(s) \) are monotone in \( s \) for \( s \) sufficiently large.
Proof: According to theorem 3.1 and its derivation, the use of the terms thru \((\log x^2)^{-r}P_r^1\) and \(\bar{x}^{-2r}\) in the expansions for \(f\) and \(\beta\) yield a discrepancy \(\Delta_{Kr}(\alpha,s) \approx s^{-1}0^*(\alpha^{-2r})\) and
\[
\frac{\partial \Delta_{Kr}}{\partial \alpha} \approx s^{-1}0^*(\alpha^{-3r}).\quad \text{Hence } \Delta_{Kr}^+(\alpha,s) \approx s^{-1}(\log s)^{-r} \quad \text{and}
\]
\[
\frac{\partial \Delta_{Kr}^+}{\partial \alpha} \approx s^{-1}0^*(\log s)^{-r-1/2} = o[s^{-1}(\log s)^{-r}].\quad \text{Applying lemma 5.2b,}
\]
\[
\hat{\alpha}_{Kr}^+ - \alpha_s = o(\log s)^{-r} \quad \text{and } \Delta_{Kr}^+(\alpha_{Kr},s) = \theta_{Kr}^+ \approx s^{-1}(\log s)^{-r}. \quad \text{The same arguments apply to } \theta_{Kr}^- \quad \text{and } \alpha_{Kr}^- \quad \text{Moreover } \alpha_{Kr}^{*+} \quad \text{exists and}
\]
\[
\alpha_{Kr}^+ - \alpha_{Kr}^- \approx (\log s)^{-r/2}, \quad \text{completing the first half of lemma 5.3.}
\]
For large \(s\),
\[
\frac{d\theta_{Kr}}{ds} = \frac{\partial}{\partial s} \Delta_{Kr}(\alpha_{Kr},s) + \frac{\partial}{\partial k} \Delta_{Kr}(\alpha_{Kr},s) \frac{\partial \alpha_{Kr}}{\partial s} = \frac{\partial}{\partial s} \Delta_{Kr}(\alpha_{Kr},s).
\]
Since \(g + Ks^{-1/2} \phi(\alpha)\) represents a solution of the heat equation, and \(s^{-1/2} (\log s)^{-r}\) corresponds to the solution of a modified version of it, it follows that \(\frac{d\theta_{Kr}}{ds} = \frac{\partial^2}{\partial \alpha^2} \Delta_{Kr}(\alpha_{Kr},s) + \frac{1}{s} = o(s^{-2})\). The asymptotic expansion for \(g\) and \(s^{-1/2} \phi(\alpha)\) used in the proof of Theorem 3.1 can be differentiated termwise to give asymptotic expansions for the partial derivatives with respect to \(\alpha\). Substituting in the above equation and canceling we get \(\frac{d\theta_{Kr}}{ds} = s^{-2}2Q_{2r'}^{-2} - 2\) for some \(r'\) unless there is complete cancellation. In both the cases of \(\theta_{Kr}^+\) and \(\theta_{Kr}^-\), cancellation must fail for some \(r' \leq r\). In each case, for \(s\) sufficiently large \(\frac{d\theta_{Kr}}{ds}\) will have only one sign determined by the polynomial \(Q_{r'}\). Clearly \(\theta_{Kr}^+\) is decreasing since it is positive and approaches zero. Similarly \(\theta_{Kr}^-\) is increasing.
Lemma 5.4. For large \( s \), \( \tilde{\alpha} \geq \alpha_r - o[\log s]^{-1/2} \).

Proof: For \( s \) sufficiently large, say \( s > s_o \), \( \vartheta^+_{or} > 0 \). Increasing \( K \) from zero, increases \( \theta^+_{kr} \), decreases \( \hat{\alpha}^+_{kr} \) and decreases \( \alpha^*_{Kr} \) wherever it exists positive. By increasing \( K \) sufficiently, we obtain \( \Delta^+_{Kr}(\alpha, s_o) > 0 \) for \( \alpha < \alpha^*_{or}(s_o) \). Hence \( \alpha^*_{Kr} \) represents the boundary of a region which corresponds to a truncated sequential procedure. In this region \( g^+_{r} + Ks^{-1/2}\varphi(\alpha) \) satisfies the heat equation. At ordinary points of the boundary \( g^+_{r} + Ks^{-1/2}\varphi = s^{-1} + s^{1/2}\varphi(\alpha) \). If, however, \( \alpha^*_{Kr} \) has discontinuities, this equality may fail. But, the fact that \( \Delta^+_{Kr} > 0 \) between \( \alpha^*_{Kr} \) and \( \hat{\alpha}^+_{Kr} \) implies that \( g^+_{r} + Ks^{-1/2}\varphi \geq s^{-1} + s^{1/2}\varphi(\alpha) \) on the boundary. Hence \( g^+_{r} + Ks^{-1/2}\varphi \) is the risk (Bayes value) corresponding to the truncated sequential procedure with a possibly increased stopping risk. Then the optimal Bayes risk \( u \) satisfies \( u(\alpha s^{1/2}, s) \leq g^+_{r} + Ks^{-1/2}\varphi \) inside the region.

Now for \( s \) sufficiently large, i.e., \( s > s(K) \), \( g^+_{r} + Ks^{-1/2}\varphi < s^{-1} + s^{1/2}\varphi(\alpha) \) for \( 0 < \alpha < \alpha^*_{Kr} \) (lemma 5.2e) and hence \( u < s^{-1} + s^{1/2}\varphi(\alpha) \) there.

But this inequality characterizes continuation points for the optimal procedure. Thus \( \tilde{\alpha} \geq \alpha^*_{Kr} \) for \( s > s(K) \). Applying lemma 5.3 yields the desired result.

Lemma 5.5. For large \( s \), \( \tilde{\alpha} \leq \alpha_r + o(\log s)^{-1} \).

Proof: Select \( s_o \) so that \( \vartheta^-(s) < 0 \) for \( s > s_o \). Decreasing \( K \) reduces \( \theta^+_{Kr} \) and increases \( \hat{\alpha}^-_{Kr} \). For \( K \) sufficiently negative, \( S = \sup_{\alpha} [g^-_{r}(\alpha, s_o) + Ks_o^{-1/2}\varphi(\alpha) - u(\alpha s_o^{1/2}, s_o)] < 0 \).
Since $\theta_{Kr}^-(s)$ is monotonic increasing (to zero) for $s$ sufficiently large, there is an $s_1$ such that

$$\theta_{Kr}^-(s_1) > S$$

and

$$\theta_{Kr}^-(s) < \theta_{Kr}^-(s_2) < 0$$

whenever $s_2 > s > s_0$ and $s_2 \geq s_1$. Now suppose that $\tilde{\alpha} > \hat{\alpha}_{Kr}^-$ for some $s_2 \geq s_1$. Then

$$h = s^{1/2} + Ks^{-1/2} \varphi(\alpha) - \theta_{Kr}^-(s_2) - u(\alpha s^{1/2}, s)$$

satisfies the heat equation in the region bounded by $s = s_o$ and $(\tilde{\alpha}(s), s)$ for $s > s_0$. Furthermore it is negative on that part of the boundary of the region corresponding to $s \leq s_2$. Thus

$$h(\hat{\alpha}_{Kr}^-(s_2), s_2) < 0,$$

and hence

$$u(\hat{\alpha}_{Kr}^-(s_2), s_2^{1/2}, s_2^{-1} + s^{1/2} \varphi(\hat{\alpha}_{Kr}^-(s_2))) > 0$$

which is impossible. Hence $\tilde{\alpha} \leq \hat{\alpha}_{Kr}^- = \alpha_r + o(\log s)^{-r}$ for large $s$ which is the desired result.

**Theorem 5.1.** $\tilde{\alpha} - \alpha_r = o \times (\log s)^{-r - 3/2}$.

**Proof:** Applying lemmas 5.4 and 5.5 $\tilde{\alpha} - \alpha_r = o(\log s)^{-r'}/2$. However

$$\alpha_{r+1} - \alpha_r = o(\log s)^{-r'/2}$$

and hence

$$\alpha_{r'} - \alpha_r = o(\log s)^{-r'/2} \quad \text{for} \quad r' > r.$$
\[ \tilde{\alpha} - \alpha_r = O(\log s)^{-r-3/2}. \]

Theorem 5.1 is the desired result that the expansion (3.14a) of Theorem 3.1 furnishes an asymptotic approximation to \( \tilde{\alpha} \). Essentially the same method could have been used with the alternative expansion of section 4.
6. Asymptotic approximation to the Bayes Risk.

A considerable portion of this section is concerned with technicalities presented in the form of lemmas. To help the reader we outline this section briefly. Goursat [6, p. 311] presents a representation of the solution of the heat equation which is equivalent to

\[
(6.1) \quad u(y_1, s_1) = \int_B \frac{1}{\sqrt{s_1-s}} \varphi\left(\frac{y_1-y}{\sqrt{s_1-s}}\right) [u(y,s)dy + \frac{1}{2} u_y ds - \frac{u}{2} \left(\frac{y_1-y}{s_1-s}\right) ds]
\]

where the integral is taken along a \((y,s)\) path \(B\) which starts at \(s=s_1\) with a value of \(y\) less than \(y_1\), passes through points with \(s<s_1\) and terminates at \(s=s_1\) with a value of \(y\) greater than \(y_1\). This representation is particularly well adapted for our problem with \(B\) consisting of a vertical section at \(s_0<s_1\) and the upper and lower optimal boundaries from \(s_0\) to \(s_1\). Along the upper and lower optimal boundaries, \(u\) and \(u_y\) are determined by the boundary conditions and coincide with \(s_1^1 + s_1^{1/2} \psi(ys^{-1/2})\) and \(1 - \psi(ys^{-1/2})\) respectively.

A minor variation of (6.1) is applied to the difference between \(u\) and \(g_\tau + Ks_1^{-1/2} \psi(\alpha)\) where \(g_\tau\) is the \(r\)th order expression in the formal expansion and \(s_0 = s_1^{1/4}\). It is required to show that the difference is small. To do so one must establish suitable bounds on various line integrals along the optimal boundary. Deriving these bounds and carrying out a number of other steps would be considerably easier if we used the symmetry of the Bayes risk and the monotonicity of the optimal boundary for our specific problem. To allow for the applicability to possibly asymmetric variations of our problem, more elaborate proofs were applied.
Lemmas 6.1 and 6.2 are used to bound integrals along the optimal boundary by similar integrals along monotone approximations to the boundary. Lemmas 6.3 and 6.4 express integrals of $u$ and $y_u$ from the lower to upper boundary in terms of $u$ and $u_y$ along the boundaries and yield the variation of the Goursat representation in lemma 6.5. Lemmas 6.6 to 6.9 consist of evaluating bounds on the various integrals in the above representation applied to $g^+ Ks^{-1/2} \varphi(\alpha) - u$. Finally we state and prove Theorem 6.1 which shows that the optimal solution is approximated by $C_0 s^{-1/2} \varphi(\alpha) + C_1 s^{-1} \varphi(\alpha) + g^+ (\alpha, s)$. The $C_1$ term is zero for our symmetric problem. This is not the case for problems with asymmetric stopping risks.

Lemma 6.1. If

1. $y_1(x)$ and $y_2(x)$ are monotonic increasing functions such that
   \[ y_1(x) < y_0(x) < y_2(x) \quad \text{for} \quad x_1 < x < x_2, \]
2. $y_1(x_1) = a_1$, $y_1(x_2) = a_2$ for $i = 0, 1, 2$, and
3. $f$ is continuous and $\frac{\partial f}{\partial x} > 0$ in the region bounded by
   \[ x_1 \leq x \leq x_2, \quad y_1(x) \leq y \leq y_2(x), \]
then

(6.2) \[ \int_{x_1}^{x_2} f[x, y_2(x)] dy_2(x) < \int_{x_1}^{x_2} f[x, y_0(x)] dy_0(x) < \int_{x_1}^{x_2} f[x, y_1(x)] dy_1(x). \]

Proof: Approximate $y_0(x)$ by a polygon which yields an approximation to the path integral. Suppose that the polygon decreases from $b_2$ to $b_1$ along one subinterval of $(x_1, x_2)$ and increases from $b_1$ to $b_2$ along a "later" subinterval. Condition (3) implies that these two
intervals contribute a positive amount to the integral. Replacing "later" by "earlier" in the first sentence leads to replacing "positive" by "negative" in the second. The repeated use of these facts lead to the desired result.

Incidently the result applies when \(y_1(x)\) and \(y_2(x)\) have vertical sections. In our applications \(y_2(x)\) will have a vertical section at \(x_1\) and \(y_1(x)\) may have one at \(x_2\).

**Lemma 6.2.** If

1. \(y_1(x)\) and \(y_2(x)\) are monotonic increasing functions such that \(y_1(x) < y_0(x) < y_2(x)\) for \(x_1 < x < x_2\),

2. \(y_1(x_1) = a_1\), \(y_1(x_2) = a_2\) for \(i=0,1,2\), and

3. \(\frac{df(x,y)}{dx} < K(y)\) in the region bounded by

   \[x_1 \leq x \leq x_2, \quad y_1(x) \leq y \leq y_2(x),\]

then

\[\int_{x_1}^{x_2} f[x,y_1(x)]dy_1(x) - \int_{x_1}^{x_2} f[x,y_0(x)]dy_0(x) \leq \int_{a_1}^{a_2} \rho(y)K(y)dy\]

where

\[\rho(y) = \sup\{|x_3-x_4|: y_1(x_3) = y_2(x_4) = y\}\]

Proof: We simply elaborate on lemma 1 using the fact that

\[|f(x_3,y) - f(x_4,y)| \leq |x_3-x_4|K(y)\]

Thus the interval \((y, y+dy)\) contributes less, in absolute value, than \(\rho(y)K(y)dy\) to the difference \(\int f[x,y_1(x)]dy_1 - \int f[x,y_0(x)]dy_0\) or to \(\int f[x,y_0(x)]dy_0 - \int f[x,y_2(x)]dy_2\).
Lemma 6.3. For a solution $u$ of the heat equation,

(6.5) $\frac{d}{ds} \int_{y_1(s)}^{y_2(s)} u(y, s) dy = \left[ \frac{1}{2} u_y(y_2, s) + u(y_2, s) \frac{dy_2}{ds} \right] - \left[ \frac{1}{2} u_y(y_1, s) + u(y_1, s) \frac{dy_1}{ds} \right]$

(6.6) $\frac{d}{ds} \int_{y_1(s)}^{y_2(s)} yu(y, s) dy = \frac{1}{2} (u(y_2, s) \left[ \frac{dy_2}{ds} \right] + y_2 u_y(y_2, s))$

$\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \q
Lemma 6.4. There is a constant $C_0$ such that the optimal risk satisfies

\[(6.7a) \quad \int_{-\gamma(s_o)}^{\gamma(s_o)} u(y,s_o)dy = C_0 - \int_{s_o}^{\infty} [u_y(\gamma,y,s)ds + 2u(\gamma,y,s)dy(s)] \]

\[(6.7b) \quad = C_0 - O[s_o^{-1/2}(\log s_o)^{1/2}] \]

Proof: Equation (6.7a) follows from equation (6.5) and symmetry provided the integral on the right converges. Along the optimal boundary, $u(\gamma,y,s) = s^{-1} + s^{1/2} \psi(ys^{-1/2})$ whose partial derivative with respect to $y$ is $-[1 - \Phi(ys^{-1/2})] = O[s^{-3/2}(\log s)^{-1/2}]$, and whose partial derivative with respect to $s$ is $-s^{-2} + \frac{1}{2} s^{-1/2} \phi(ys^{-1/2})$. Using the first few terms of the asymptotic expansion

\[\alpha^2 = \gamma^2 s^{-1} \approx 3 \log s - \log 8\pi - 2(\log s)^{-1} + \ldots\]

it follows that the above partial derivative with respect to $s$ is positive for $s$ sufficiently large when $\alpha$ is between the lower and upper approximations obtained by changing the coefficient of $(\log s)^{-1}$ above to $-3$ and $-1$ respectively. Now we may apply lemma 6.3 to

\[\int_{s_o}^{\infty} u(\gamma,y,s)dy(s) = \int_{s_o}^{\infty} [s^{-1} + s^{1/2} \psi(\gamma s^{-1/2})]dy(s)\]

using these monotonic lower and upper approximations to the boundary.

Then

\[\int_{s_o}^{\infty} u(\gamma,y,s)dy(s) = O[s_o^{-1/2}(\log s_o)^{1/2}]\]

and

\[\int_{s_o}^{\infty} u_y(\gamma,y,s)ds = - \int_{s_o}^{\infty} [1 - \Phi(\gamma s^{-1/2})]ds = O[s_o^{-1/2}(\log s_o)^{-1/2}]\]
and hence the integral in equation (6.7a) converges and equation (6.7b) follows.

Lemma 6.4 and its method of proof can be pursued further with profit. For example an analysis of the optimal procedure at \( s = 0 \) shows that the integral on the left of (6.7a) approaches zero as \( s \to 0 \). Hence

\[
C_o = \int_{B^o} \frac{1}{2} u_y(\tilde{y}, s) ds + u(\tilde{y}, s) d\tilde{y}(s)
\]

where \( B^o \) is the path described by the optimal boundary going from the lower boundary and \( s = \infty \) to \( s = 0 \) and then along the upper boundary to \( s = \infty \). This result is also valid in the asymmetric case.

Finally, lemma 6.4 depends mainly on the convergence of the integral on the right of (6.7a). Thus equations (6.7a) and (6.7b) are valid if \( u \) is replaced by \( g_1 + K_0 s^{-1/2} \phi(\alpha) + K_1 s^{-1} \alpha \phi(\alpha)^{-1} u \) and \( C_o \) is replaced by \( C_{or} + K_0 - C_o \).

In the asymmetric case an analogue of lemma 6.4 concerning \( \int y u d\tilde{y} \) would become relevant. This analogue would involve

\[
C_1 = \int_{B^o} \frac{1}{2} (u(\tilde{y}, s)[d\tilde{y}^2(s) - ds] + \tilde{y} u_y(\tilde{y}, s) ds.
\]

This analogue applied to \( g_1 u \) appears in the proof of lemma 6.6 (see eq. 6.19).

We now apply lemma 6.4 to the Goursat representation (6.1) to derive the following minor variation of the Goursat representation using sections of the optimal boundary and a vertical section for the path \( B \).
Lemma 6.5. For \((y_1, s_1)\) within the optimal continuation region, the optimal risk satisfies

\begin{equation}
(6.10) \quad u(y_1, s_1) = c_o s_1^{-1/2} \varphi(y_1 s_1^{-1/2}) + \int_{-\tilde{y}(s_o)}^{\tilde{y}(s_o)} \left\{ \frac{1}{\sqrt{s_1 - s_0}} \varphi \left( \frac{y_1 - y}{\sqrt{s_1 - s}} \right) - \frac{1}{\sqrt{s_1}} \varphi \left( \frac{y_1}{\sqrt{s_1}} \right) \right\} u(y, s_o) \, dy
\end{equation}

\[ + \int_{s_o}^{s_1} \left\{ \frac{1}{\sqrt{s_1 - s}} \varphi \left( \frac{y_1 - \tilde{y}}{\sqrt{s_1 - s}} \right) + \varphi \left( \frac{y_1 + \tilde{y}}{\sqrt{s_1 - s}} \right) - \frac{2}{\sqrt{s_1}} \varphi \left( \frac{y_1}{\sqrt{s_1}} \right) \right\} \left\{ u(\tilde{y}, s) \, d\tilde{y} + \frac{1}{2} u_y(\tilde{y}, s) \, ds \right\} \]

\[ + \int_{s_o}^{s_1} \frac{1}{2(s_1 - s)^{3/2}} \left\{ \left( y_1 + \tilde{y} \right) \varphi \left( \frac{y_1 + \tilde{y}}{\sqrt{s_1 - s}} \right) - \left( y_1 - \tilde{y} \right) \varphi \left( \frac{y_1 - \tilde{y}}{\sqrt{s_1 - s}} \right) \right\} \]

\[ u(\tilde{y}, s) \, ds \]

\[ - \frac{1}{\sqrt{s_1}} \varphi \left( \frac{y_1}{\sqrt{s_1}} \right) \int_{s_1}^{\infty} \left\{ 2u(\tilde{y}, s) \, d\tilde{y} + u_y(\tilde{y}, s) \, ds \right\} , \quad s_0 < s_1 \]

The same equation applies to \(u_r(y, s) = g_r(ys^{-1/2}, s)\) with \(c_o\) replaced by \(c_{or}\).
By applying equation (6.10) to \( u_r - u \), we shall show that

\[
(6.11) \quad u(y, s) = c_0 s_1^{-1/2} \varphi(\alpha) + g_r(\alpha, s) + s_1^{-1} 0^{*} [(\log s)^{-R - 1}]
\]

To accomplish this we require approximations and bounds for the various integrals appearing in (6.10). In order to avoid arguments which make essential use of the symmetry in our particular problem, some of the ensuing discussion will be considerably longer than necessary.

**Lemma 6.6.** Let \( s_0 = s_1^{1/4} \). Then for \( r > 1 \)

\[
(6.12) \quad I_1 = \int_{\tilde{y}(s_0)}^{\tilde{y}(s_0)} \left\{ \frac{1}{\sqrt{s_1 - s_0}} \varphi \left[ \frac{y_1 - y}{\sqrt{s_1 - s_0}} \right] - \frac{1}{\sqrt{s_1}} \varphi \left[ \frac{y_1}{\sqrt{s_1}} \right] \right\} [u_r(y, s_0) - u(y, s_0)] dy
\]

\[= c_{l_1 r_1} s_1^{-1} \alpha \varphi(\alpha) + 0^{*} [s_1^{-1} (\log s_1)^{-R + 1}].\]

where \( c_{l_1 r_1} = 0 \) for a symmetric stopping risk.

**Proof:**

\[
(6.13) \quad A_1 = \frac{1}{\sqrt{s_1 - s_0}} \varphi \left( \frac{y_1}{\sqrt{s_1 - s_0}} \right) - \frac{1}{\sqrt{s_1}} \varphi \left( \frac{y_1}{\sqrt{s_1}} \right) = 0(s_0 s_1^{3/2}).
\]

Since \( u_r - u \) is bounded

\[
(6.14) \quad \int_{\tilde{y}(s_0)}^{\tilde{y}(s_0)} A_1 [u_r(y, s_0) - u(y, s_0)] dy = 0[s_0^{3/2} (\log s_0)^{1/2} s_1^{-3/2}]
\]

\[= 0(s_1^{-9/8} (\log s_1)^{1/2}).\]
(6.15) \[ A_2 = \frac{1}{s_1^{1/2}} \left[ \varphi \left( \frac{y_1 - y}{s_1} \right) - \varphi \left( \frac{y_1}{s_1} \right) \right] - y_1 - 1 \alpha_1 \varphi(\alpha_1) + y_2^2 \left( s_1^{3/2} \right) . \]

First we note that

(6.16) \[ \int_{\tilde{y}(s)}^{\tilde{y}(s)} y^2 [u_r(y, s_0) - u(y, s_0)] \, dy = 0 \left[ s_0^{3/2} (\log s_0)^{3/2} \right] = 0 \left[ s_1^{3/8} (\log s_1)^{3/8} \right] \]

This together with (6.14) and the fact that \( \int_{\tilde{y}(s)}^{\tilde{y}(s)} y (u_r - u) \, dy = 0 \) in our symmetric problem establishes the lemma with \( C_r = 0 \). However, for asymmetric versions of the problem, the latter integral does not necessarily vanish. For a proof which applies to such versions, we apply equation (6.6) to transform the integral \( \int y (u_r - u) \, dy \) to one with respect to \( s \). Let \( y_1(s) \) be the lower optimal boundary \( -\tilde{y}(s) \) and \( y_2(s) \) be the upper optimal boundary \( \tilde{y}(s) \). Then along these optimal boundaries

(6.17a) \[ u_r - u = [g_r + K s^{-1/2} \varphi(\alpha)] - [s^{-1} + s^{1/2} \varphi(\alpha)] = \Delta (y_1^{-1/2}, s) \]

and

(6.17b) \[ (u_r - u) \right |_y = s^{-1/2} \frac{\partial \Delta}{\partial \alpha} . \]

Thus we are interested in

(6.18) \[ \int_{s_0}^{\infty} \Delta \cdot [d\tilde{y}^2 \, ds] + y_1^{-1/2} \frac{\partial \Delta}{\partial \alpha} \, ds \]

Using the results of section 5 it is clear that there exist approximations \( y_{3r}(s) \) and \( y_{4r}(s) \), \( r > 1 \), such that

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(i) \( \tilde{y}_{3r}(s) < \tilde{y}(s) < \tilde{y}_{4r}(s) \)

(ii) \( \tilde{y}_{3r}(s) \) and \( \tilde{y}_{4r}(s) \) are monotonic for \( s \) large

(iii) \( \Delta(\alpha, s) = s^{-1}0^*[\log(s)]^{-r-1} \) along \( \tilde{y}(s) \) and these approximations,

(iv) \[ \frac{\partial \Delta}{\partial \alpha} = s^{-1}0^*[\log(s)]^{-r-\frac{3}{2}} \]

(v) \[ \frac{\partial \Delta(y_{s}^{-1/2}, s)}{\partial s} = (2s)^{-1} \frac{\partial^2 \Delta}{\partial \alpha^2} + s^{-2} = o(s^{-2}) \]

(iv) \( \tilde{y}_{4r}^2 - \tilde{y}_{3r}^2 = 0*[s(\log s)^{-r}] \).

Then we may apply lemma 6.2 to \( \int_{s_0}^{\infty} \Delta \, d\tilde{y}^2(s) \) where the \( \tilde{y}_0, \tilde{y}_1, \tilde{y}_2 \) of lemma 6.2 are replaced by \( \tilde{y}_{3r}, \tilde{y}^2 \) and \( \tilde{y}_{4r}^2 \) except that \( \tilde{y}_{3r}^2 \) and \( \tilde{y}^2 \) are modified with short vertical sections at \( s_0 \) so that the three functions coincide at \( s_0 \). The role of the function \( \rho(y) \) of lemma 6.2 is taken by

\[ \rho[y^2] = 0*[s(\log s)^{-r-1}] = 0*[y^2(\log y^2)^{-r-2}] \]

and that of the function \( K(y) \) of lemma 2, is taken by

\[ K(y^2) = o(s^{-2}) = o[(y^2)^{-2}(\log y^2)^2] \]

Then the integral

\[ \int_{\tilde{y}(s_0)}^{\infty} \rho(y^2)K(y^2) \, dy^2 = o[\log \tilde{y}_{4r}^2(s_0)]^{-r+1} = o[\log s_1]^{-r+1} \]

Hence
\[ \int_{\infty}^{s_0} \Delta \, dy^2(s) \leq \int_{\infty}^{s_0} \Delta \, dy_{lr}^2(s) + o(\log s_1)^{-r+1} \]

\[ = \int_{\infty}^{s_0} s^{-1} 0*(\log s)^{-r-1} \, [\log s] \, ds + o(\log s_1)^{-r+1} \]

\[ = 0*[(\log s)^{-r+1}]. \]

Furthermore

\[ \int_{\infty}^{s_0} \Delta \, ds = 0*[(\log s_1)^{-r}] \]

and

\[ \int_{\infty}^{s_0} \tilde{y}s^{-1/2} \, \frac{d\Delta}{d\alpha} \, ds = 0*[(\log s_1)^{-r}] . \]

Thus the expression in (6.18) is $0*[(\log s)^{-r+1}]$ and the application of equation (6.6) has yielded

(6.19) \[ \int_{\infty}^{s_0} \tilde{y}(s_0) \frac{\tilde{y}[u_x(y, s_0) - u(y, s_0)]}{-\tilde{y}(s_0)} \, dy = C_{lr} + 0*[(\log s_1)^{-r+1}] \]

Combining (6.14)-(6.17) with (6.19) we have

\[ I_1 = 0[s_1^{-9/8}(\log s_1)^{1/2} + 0[s_1^{-3/2+3/8}(\log s_1)^{3/8}] \]

\[ + \alpha_1^{\varphi}(\alpha_1)s_1^{-1}[c_{lr} + 0*[(\log s_1)^{-r+1}]] \]

which gives the result claimed in lemma (6.6). For the symmetric case $C_{lr} = 0$. For asymmetric stopping risks

(6.20) \[ c_{lr} = \frac{1}{2} \int_{B_0} \Delta \, [dy^2 - ds] + \tilde{y}s^{-1/2} \, \frac{d\Delta}{d\alpha} \, ds . \]
Lemma 6.7. Let \( s_0 = s_1^{1/4} \), \(|y_1| \leq \sqrt{2.5s_1}\log s_1\) and \( r > 1 \). Then

\[
I_2 = \int_{s_0}^{s_1} \left\{ \frac{1}{\sqrt{s_1-s}} \varphi \left( \frac{y_1-\tilde{y}}{\sqrt{s_1-s}} \right) + \varphi \left( \frac{y_1+\tilde{y}}{\sqrt{s_1-s}} \right) - \frac{2}{\sqrt{s_1}} \varphi \left( \frac{y_1}{\sqrt{s_1}} \right) \right\}
\]

\[
\left\{ \left[ u_r(\tilde{y},s) - u(\tilde{y},s) \right] d\tilde{y}(s) + \frac{1}{2} \left[ u_{ry}(\tilde{y},s) - u_y(\tilde{y},s) \right] ds \right\}
\]

\[
= s_1^{-1} O^*[(\log s_1)^{-r+1}].
\]

Proof: Once more we avoid the use of symmetry. Applying the condition on \( y_1 \), to \( s_0 < s < s_1 \) with \( y = y(s) = \sqrt{3s \log s} \), we obtain

\[
(6.21) \quad \frac{1}{\sqrt{s_1-s}} \varphi \left( \frac{y_1+\tilde{y}}{\sqrt{s_1-s}} \right) - \frac{1}{\sqrt{s_1}} \varphi \left( \frac{y_1}{\sqrt{s_1}} \right) = 0[s_1^{-3/2} + O[y_1^{-1}]
\]

\[
= O[s_1^{-1}s^{-1/2}(\log s_1)^{1/2}].
\]

Along the optimal boundary \( u_r-u \) and \( u_{ry}-u_y \) coincide with \( \Delta \) and \( s^{-1/2}\partial\Delta/\partial\alpha \).

\[
(6.22) \quad u_r(\tilde{y},s) - u(\tilde{y},s) = \Delta = s_1^{-1} O^*[(\log s_1)^{-r-1}]
\]

and

\[
(6.23) \quad u_{ry}(\tilde{y},s) - u_y(\tilde{y},s) = s^{-1/2} \partial\Delta/\partial\alpha = s^{-3/2} O^*[(\log s_1)^{-r-3/2}].
\]

That part of \( I_2 \) contributed by \( u_{ry}-u_y \) is easily seen to be

\[
s_1^{-1} O^*[(\log s_1)^{-r}].
\]

On the other hand the remaining part of \( I_2 \) requires the application of lemma 6.2. The \( y_o, y_1, \) and \( y_2 \) of lemma 6.2 are \( \tilde{y}, y_{3r} \) and \( y_{4r} \) respectively. The distance function corresponding to \( \rho \) is now
(6.24) \[ \rho(y) = 0^* [s(\log s)^{-r-1}] \]

where \( s \sim y^2 / \log y^2 \).

Now we compute a bound for \( K(y) \), the derivative with respect to \( s \) of the integrand which we identify with (6.21) multiplied by \( \Delta(ys^{-1/2}, s) \). The derivative of (6.21) with respect to \( s \) (keeping \( y \) fixed) is

\[
\frac{1}{2(s_1-s)^{3/2}} \left\{ \left[ 1 - \frac{(\frac{y}{s_1-s})^2}{s_1-s} \right] \phi \left( \frac{y+s}{\sqrt{s_1-s}} \right) \right\} = 0\left[ s_1^{-3/2} \right].
\]

Since \( \Delta \) is a solution of the heat equation, the derivative of \( \Delta \) with respect to \( s \), keeping \( y \) fixed is \( (2s)^{-1} [\partial^2 \Delta / \partial \alpha^2] \). Applying lemma 5.2b, the required derivative is \( 0(s^{-2}) \). Hence

\[ K(y) = 0^* [s_1^{-3/2} s_1^{-1} (\log s_1)^{-r-1}] + 0[s_1^{-1} s_1^{-3/2} (\log s_1)^{1/2}] \]

and

\[
\int_{y^{br}(s_1)}^{y^{3r^c}(s_1)} \rho(y) K(y) dy = s_1^{-1} 0^* (\log s_1)^{-r+1}.
\]

Then lemma 6.2 yields \( s_1^{-1} 0^* (\log s_1)^{-r+1} \) for that part of \( I_2 \) contributed by \( (u_{ry}-u_y) \) terms, and lemma 6.7 follows.

\textbf{Lemma 6.8.} Let \( s = s_1^{1/4} \) and \( |y| \leq \sqrt{2.5 s_1 \log s_1} \). Then

\[
I_3 = \int_{s_0}^{s_1} \frac{1}{2(s_1-s)^{3/2}} \left\{ (y_1+\bar{y}) \phi \left( \frac{y_1+\bar{y}}{\sqrt{s_1-s}} \right) \right\} [u_r(\bar{y}, s)-u(y, s)] ds
\]

\[ = 0\left[ s_1^{-1} (\log s_1)^{-r} \right] \]

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Proof: The result follows from the fact that the integrand is bounded by

\[ s_1^{1/2} s_1^{-1} o^*[(\log s_1)^{-r-1}] \]

Lemma 6.9.

\[ I_4 = \int_{s_1}^{\infty} [2[u_{\gamma}(\tilde{y},s) - u(\tilde{y},s)] d\tilde{y}(s) + [u_{ry}(\tilde{y},s) - u_y(\tilde{y},s)] ds] 
   = s_1^{-1/2} o^*[(\log s_1)^{-r+1/2}]. \]

Proof: The contribution of \( u_{ry} - u_y \) is \( s_1^{-1/2} o^*[(\log s_1)^{-r+3/2}] \). The other part may be bounded using some of the computations of lemma 9.8 in the application of lemma 6.2. Using the same \( \tilde{y}, y_{3r}, y_{4r} \) we have the same \( \rho(y) \), (see eq. 6.24). On the other hand the new \( K(y) \) is \( O(s^{-2}) \) and \( \int K \rho dy = O[s_1^{-1/2}(\log s_1)^{-r-1/2}] \). The approximation using \( y_{3r} \) gives an amount equal to \( s_1^{-1/2} o^*[(\log s_1)^{-r+1/2}] \). Combining these results we have our lemma.

Theorem 6.1. The formal expansion of Theorem 3.1 provides an asymptotic approximation to the optimal risk \( u(y,s) \) in the following sense. For \( (y,s) \) within the optimal continuation region and \( r > 1 \),

\[ u(y,s) = C_0 s_1^{-1/2} \phi(\alpha) + C_1 s_1^{-1} \alpha \phi(\alpha) + g_T(\alpha,s) + s_1^{-1} o^*[(\log s)^{-r-1}] \]

where

\[ C_0 = \int_B \frac{1}{2} u_y(\tilde{y},s) ds + u(\tilde{y},s)d\tilde{y}(s) \]

is the line integral along the optimal boundary and \( C_1 = 0 \) in our symmetric problem.
Proof: Assembling lemmas 6.6-6.9, we have for $|y_1| \leq \sqrt{2.5s_1 \log s_1}$

\[(6.27) \quad u(y_1,s_1) = C_0 s_1^{-1/2} \varphi(\alpha) + g_r(\alpha_1,s) + s_1^{-1/2} [\log s_1]^{-r+1}] .\]

The proofs of Lemmas 6.7 and 6.8 were not quite delicate enough to obtain the same result for $|y_1| \geq \sqrt{2.5s_1 \log s_1}$. However in this range we know from the proof of lemma 5.4 that for some $K$,

$u \leq g_{Kr} + Ks_1^{-1/2} \varphi(\alpha) = O(s_1^{-1})$ and $u_r = O(s_1^{-1})$. Hence

$v(y,s) = g_r(\alpha_1,s) + C_0 s_1^{-1/2} \varphi(\alpha) - u(y,s) = O(s_1^{-1/2})$

satisfies the heat equation in the region bounded by $s_1/12 \leq s \leq s_1$ and the optimal boundaries. Hence [4],

$v(y,s) = \mathbb{E}[v(Y,S)]$

where $(Y,S)$ is the random point where a Wiener Process going backwards in the $s$ scale from the point $(y_1,s_1)$ first intersects the boundary of the above region. But $v = s_1^{-1/2} [\log s_1]^{-r+1}$ along the optimal boundaries. Hence these sections contribute $s_1^{-1/2} [\log s_1]^{-r+1}$ to $v$. Along the section where $s = s_1/12$, $v = O(s_1^{-1})$. The probability that the Wiener process from $(y_1,s_1)$ with $y_1 \geq \sqrt{2.5s_1 \log s_1}$ intersects this vertical section is bounded by

$\phi\left\{ \frac{[2.5s_1 \log s_1]^{1/2} - [3(s_1/12) \log(s_1/12)]^{1/2}}{[11s_1/12]^{1/2}} \right\} = O(s_1^{-5/4}) .$

Hence the vertical section of the boundary contributes $O(s_1^{-1.04})$ to $v$ and $v = s_1^{-1/2} [\log s_1]^{-r+1}$ for $|y_1| \geq [2.5s_1 \log s_1]^{1/2}$. Combining
this with (6.27) we have the desired result except that the exponent
\(-r-1\) of \(\log s_1\) is replaced by \(-r+1\). However, since
\(e_{T+2} - e_r = s_1^{-1} 0^*[(\log s_1)^{-r-1}]\), the exponent can be improved to \(-r-1\)
by first applying the weaker result to \(r^* = r+2\).
7. **Bounds and Motivation.**

The original motivation for the expansions of section 3 came from the application of the Goursat formula (6.1). Note that in the second integral of (6.10) if we set \( y_1/\sqrt{s_1} = \alpha_1 \) and \( \tilde{y}/\sqrt{s_1} = b \), part of the integrand is approximately \( \varphi(\alpha+b) + \varphi(\alpha-b) - 2\varphi(\alpha) \). However the original application of the Goursat formula depended heavily on knowing that \( u \approx s^{-1} \) on the boundary. To ascertain this it was necessary to obtain bounds for the boundary to show that

\[ \tilde{y} \approx \sqrt{s}[3 \log s + o(1)] \]

Some of the methods used to obtain bounds seem to be of interest and will be sketched very briefly here.

a) **Upper bound for optimal boundary.**

We use the monotonicity of the upper boundary (in our special problem) and suppose that it passes through \((y_0, s_0)\). We obtain a bound on the Bayes risk \( b(y_1, t_1) \) where \( y_1 < y_0 \) and \( s_1^{-} = t_1^{-} > s_0^{-} = t_0^{-} \).

\[ d(y_1, t_1) = s_1^{1/2} \psi(y_1/\sqrt{s_1}) \geq b(y_1, t_1) \geq (t_0^{-}, t_1^{-}) P \]

where \( P \) is the probability that a Wiener process \( Y_s \) through \((y_1, s_1)\) going backwards in \( s \) does not cross the boundary between \( s_0 \) and \( s_1 \). But

\[ 1-P \leq \left\{ \sup_{s_0 < s < s_1} Y_s > y_0 \right\} \]

By the reflection principle

\[ 1-P \leq 2\varphi \left( \frac{y_1-y_0}{\sqrt{s_1-s_0}} \right) \]

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Let

\[ y_1 - y_0 = -\sqrt{\frac{2s_0}{3 \log s_0}} \]

\[ s_1 - s_0 = \frac{2s_0}{3 \log s_0} . \]

Then

\[ \psi \left[ \frac{y_0 - \sqrt{\frac{2s_0}{3 \log s_0}}}{s_0(1 + \frac{2}{3 \log s_0})} \right] \geq \frac{2s_0^{-3/2}}{3 \log s_0} \left[ 1 + \frac{2}{3 \log s_0} \right]^{-3/2} [1 - 2\delta(1)] \]

from which it follows that

\[ (7.1) \quad y_0^2 \leq s_0 [3 \log s_0 + O(1)] . \]

The main point of the derivation is that if the boundary is too high at time \( t_0 \), the expected cost of sampling at time \( t_1 \) is higher than the cost of stopping and deciding. The derivation can be refined to get specific constants to replace the \( O(1) \) term but I do not believe that \( -\log 8\alpha \) can be attained this way. In fact it is remarkable how effective this rather naive approach is.

b) A weak lower bound for the optimal boundary.

In this subsection we study the procedure which consists of terminating when there is no fixed sample time procedure which does as well as stopping. Obviously the boundary for this procedure is below the optimal boundary. The bound obtained by this argument is not strong enough to show that \( \bar{y} \geq \sqrt{s(3 \log s + O(1))} \) but the argument
presents a simple use of a technique which can be extended to obtain that result. This technique consists of using the identity.

\[(7.2) \quad \tilde{d}(y_o, t_o) - \int_{-\infty}^{\infty} \tilde{d}(y_o + \varepsilon, t_{1}) \varphi(\varepsilon) d\varepsilon = \gamma \psi(y_o / \gamma) \text{ for } t_o < t_{1}\]

where

\[(7.3) \quad \gamma^2 = t_o^{-1 - t_{1}^{-1}}\]

and \( \tilde{d} \) represents the stopping risk (without the sampling cost) i.e.,

\[\tilde{d}(y, t) = t^{-1/2} \psi(y \sqrt{t}) .\]

The identity (7.2) is easily derived using the relations

\[(7.4) \quad \tilde{d}^+(y_o, t_o) = \int_{-\infty}^{\infty} \tilde{d}^+(y_o + \varepsilon, t_{1}) \varphi(\varepsilon) d\varepsilon ,\]

\[(7.5) \quad \tilde{d}^+(y, t) - \tilde{d}^-(y, t) = -y ,\]

and

\[(7.6) \quad \tilde{d}(y, t) = \tilde{d}^+(y, t) \quad \text{for } y > 0 \]

\[= \tilde{d}^-(y, t) \quad \text{for } y < 0\]

where \( \tilde{d}^+ \) and \( \tilde{d}^- \) are the stopping costs associated with deciding that the mean is positive and negative respectively.

For the procedure which consists of stopping when there is no fixed sample time which does as well as stopping, the boundary is given by \( y^*(t) \) where

\[(7.7) \quad \tilde{d}[y^*(t_o), t_o] = \inf_{t_{1} > t_o} \left[ (t_{1} - t_o) + \int_{-\infty}^{\infty} \tilde{d}[y^*(t_o) + \varepsilon, t_{1}] \varphi(\varepsilon) d\varepsilon \right] \]

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(7.8) \[ d(y^*(t_0), t_0) = \inf_{t_0 > t_0} \left\{ [t_1 - t_0] + d[y^*(t_0), t_0] - \gamma \psi[y^*(t_0)/\gamma] \right\}. \]

(7.9) \[ \inf_{t_1 > t_0} \left\{ (t_1 - t_0) - \gamma \psi[y^*(t_0)/\gamma] \right\} = 0, \quad \gamma^2 = t_0^{-1} - t_1^{-1}. \]

Setting the derivative with respect to \( t_1 \) equal to zero we have two determining relations.

(7.10a) \[ \psi[y^*(t_0)/\gamma] = 2\gamma t_1^2 \]

(7.10b) \[ \psi[y^*(t_0)/\gamma] = (t_1 - t_0)/\gamma \]

Take \( t_1/t_0 = u \) large, and substituting in the above relations, we derive

\[ 2u \approx [y^*(t_0)/\gamma]^2 = -3 \log t_0 - 4 \log[-\log t_0] - \log(81 \pi/2) + o(1), \]

(7.11) \[ y^*(t_0)^2 = t_0^{-1} [-3 \log t_0 - 4 \log[-\log t_0] - \log(81 \pi e^2/2)] + o(1). \]

Note how \( t_0 y^*(t_0) \) compares with \( \tilde{a}(t_0) = -3 \log t - \log 8 \pi + o(1) \), and that at the \( y^* \) boundary,

\[ d[y^*(t_0), t_0] \approx \left( \frac{\gamma^2}{2} \right) t_0 (-\log t_0) \]

Incidentally, if \( y^* \) is replaced by \( y \), the right hand side of (7.8) represents the risk for the best fixed sample size procedure.

For \( \gamma = 0 \), a simple computation yields a risk of approximately

\[ ((8\pi)^{1/4} + (8\pi)^{-1/4}) t_0^{1/4} \] for \( t_0 \) small and \( t_0^{-1/2} \) for \( t_0 \) large.
8. The Wald Procedure.

For comparison purposes we derive an asymptotic approximation to the boundary and risk for the best Wald procedure when \( t \) is small. By this we mean, the following: A Wald procedure consists of stopping when \(|X| \geq a\). If \( \mu \) has an a priori normal distribution with mean 0 and variance \( t^{-1} \), the best \( a \) will be denoted by \( a_o = \alpha_o t^{-1/2} \) and the corresponding Bayes risk by \( R_o(\alpha, t) \). In [2], the following expressions were derived and applied.

\[
(8.1) \quad a_o^3 = \int_0^\infty v^2 \text{sech}^2 v e^{-v^2/2\alpha_o^2} \, dv / \int_0^\infty [\text{sech}^2 v + v^{-1}\text{tanh} v] e^{-v^2/2\alpha_o^2} \, dv
\]

\[
(8.2) \quad R_o(\alpha, t) = (2\pi \alpha_o^2 a_o^2)^{-1/2} \int_0^\infty [v[1-\text{tanh} v] + 2a_o^3 \frac{\text{tanh} v}{v}] e^{-v^2/2\alpha_o^2} \, dv.
\]

We shall use them to derive

**Theorem 8.1.** As \( t \rightarrow 0 \)

\[
(8.3) \quad \alpha_o = t^{-1/2} (-\log t)^{-1/3} (2c_2)^{1/3} [1+o(1)]
\]

and

\[
(8.4) \quad R_o(\alpha, t) = (2\pi)^{-1/2} t^{1/2} (-\log t)^{2/3} (2c_2)^{-2/3} (c_2^2 + 2c_2) [1+o(1)],
\]

where

\[
(8.5) \quad c_2 = \int_0^\infty v^2 \text{sech}^2 v \, dv \approx .75
\]

and

\[
(8.6) \quad c_3 = \int_0^\infty v[1-\text{tanh} v] \, dv \approx .4
\]

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Proof: The proof we present can be extended to yield more refined approximations involving

\[(8.7) \quad c_1 = \int_0^\infty \text{sech}^2 v \, dv = 1\]

and

\[(8.8) \quad c_4 = \int_0^\infty v^{-1} \left( \tanh v + e^{-2v/2} - 1 \right) dv \approx 0.95.\]

Expanding \(\exp(-v^2/2\alpha^2)\) in a Taylor expansion and noting that

\[e^{-x^2} - 1 + x - \frac{x^2}{2} + \frac{(-1)^{n-1} x^n}{n!} x^{-(n+1)}\]

is bounded for \(0 < x < \infty\), one immediately obtains

\[(8.9) \quad \int_0^\infty \text{sech}^2 v \, e^{-v^2/2\alpha^2} \, dv = c_1 + O(\alpha^{-2}),\]

\[(8.10) \quad \int_0^\infty v^2 \text{sech}^2 v \, e^{-v^2/2\alpha^2} \, dv = c_2 + O(\alpha^{-2}),\]

and

\[(8.11) \quad \int_0^\infty v[1-\tanh v]e^{-v^2/2\alpha^2} \, dv = c_3 + O(\alpha^{-2}).\]

We now proceed to establish

\[(8.12) \quad I = \int_0^\infty \frac{1}{v} \text{tanh} v \, e^{-v^2/2\alpha^2} \, dv = \log \alpha + c_4 + O(\alpha^{-2}).\]

First we note that \(I = I_1 + I_2 + c_4\) where

\[I_1 = \int_0^\infty v^{-1} \left[ \tanh v - 1 \right] e^{-v^2/2\alpha^2} \, dv = O(\alpha^{-2})\]

and
\[ I_2 = \int_0^\infty v^{-1} \left[ e^{-v^2/2\alpha_0^2} - e^{-v^2/2} \right] dv, \]

\[ = \int_0^1 v^{-1} \left[ e^{-v^2/2\alpha_0^2} - 1 \right] dv - \int_0^1 v^{-1} \left[ e^{-v^2/2} - 1 \right] dv + \int_1^\infty v^{-1} e^{-v^2/2\alpha_0^2} dv, \]

\[ - \int_1^\infty v^{-1} e^{-v^2/2} dv, \]

\[ = \int_0^{\alpha_0} v^{-1} \left[ e^{-v^2/2} - 1 \right] dv - \int_0^{\alpha_0} v^{-1} e^{-v^2/2} dv + \int_1^{\alpha_0} v^{-1} e^{-v^2/2} dv, \]

\[ = \int_1^{\alpha_0} v^{-1} \left[ e^{-v^2/2} - 1 \right] dv - \int_1^{\alpha_0} v^{-1} e^{-v^2/2} dv = \log \alpha_0, \]

and (8.12) follows. Substituting (8.9)-(8.12) into (8.1) and (8.2) we have

\[ a_0^3 = c_2 \left[ \log \alpha_0 + (c_1 + c_4) \right]^{-1} [1 + o(\alpha_0^2)] \]

and

\[ R_0(t) = (2\pi)^{-1/2} t^{1/2} \left[ \log \alpha_0 + (c_1 + c_4) \right]^{2/3} c_2^{-2/3} [c_2 + 2c_2] [1 + o(1)]. \]

Combining (8.13) with \( \alpha_0 = a_0 t^{-1/2} \) we have

\[ 3 \log \alpha_0 = -\frac{3}{2} \log t \log [-\log t] + \log 2c_2 + o(1) \]

from which (8.3) and (8.4) follow.
As is to be expected both $\alpha_0$ and $R_0(o,t)$ are larger than $\tilde{\alpha}$ and $R_0(o,t)$. What is not so expected is that they would be larger by an order of magnitude. However this order of magnitude is determined by $(-\log t)^{2/3}$ and for practical considerations it is worth remarking that this term increases very slowly as $t \to 0$. 
9. **Sampling cost and error cost.**

In sections 3 and 4 asymptotic expansions were presented for the Bayes risk and the optimal boundary as \( t \to 0 \). It is of some interest to decompose the Bayes risk into two parts; one corresponding to the sampling cost or expected time of sampling and the other to the cost due to the possibility of coming to the wrong conclusion.

Let \( u_1 \) represent the sampling cost (including the time necessary to go from 0 to \( t=s^{-1} \)), and let \( u_2 \) represent the cost due to error. Then both \( u_1 \) and \( u_2 \) satisfy the heat equation subject to the boundary conditions \( u_1 = s^{-1} \) and \( u_2 = s^{-1/2} \psi(\alpha) \) respectively. Since \( u_1 + u_2 = u \) the derivation of an asymptotic expansion for \( u_1 \) will suffice to yield one for \( u_2 \). To derive an expansion for \( u_1 \) we may substitute in the formal expansions of sections 3 or 4 and apply the expansion for the optimal boundary.

Thus one easily derives an expansion of the form

\[
    u_1 \approx k_1 s^{-1/2} \varphi(\alpha) + \int_0^b [\varphi(\alpha+b)+\varphi(\alpha-b)-2\varphi(\alpha)]f_1(s^{1/2}b)db
\]

where \( f_1 \) is an expansion determined by the expansion for \( \alpha \) on the optimal boundary and

\[
    D^*\left[f_1\left(s^{1/2}r_s(\alpha)\right)\right] = s^{-1} .
\]

Substitution in the expansion of section 4 yields

\[
    f_1(x) \approx f_0^*(x) + c_{11}x^{-2} + c_{21}x^{-4}[f_0^*(x)]^{-1} + c_{31}x^{-6}[f_0^*(x)]^{-2} + \ldots .
\]

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As in section 6, the constant $K_1$ may be expressed in terms of the integral

$$K_1 = \int_{B_0} \left[ \frac{1}{2} u_{1y}(\tilde{y}, s) ds + u_1(\tilde{y}, s) d\tilde{y} \right].$$

When $t$ is small ($s$ large), $u_1(\tilde{y}, s) \approx s^{-1}$ and $u_{1y}(\tilde{y}, s) \approx s^{-1/2}E[s^{1/2}(\alpha + \epsilon)]$. To evaluate $K_1$ seems to require the numerical solution of the heat equation for the region determined by the optimal boundary and the boundary condition $u_1 = s^{-1}$.

Applying the methods of section 4, we compute the coefficients

$$c_{11} = -3, \; c_{21} = 3, \; c_{31} = -27/2, \; c_{41} = -351/2, \; c_{51} = -6705/4, \; c_{61} = -592119/20.$$  

Consequently the coefficients corresponding to

$$f_2 = c_{12} x^{-2} + c_{22} x^{-4} [f_0(x)]^{-1} + c_{32} x^{-6} [f_0(x)]^{-2} + ...$$

which generates the expansion for $u_2$ are

$$c_{12} = 2, \; c_{22} = -6, \; c_{32} = 24, \; c_{42} = -114, \; c_{52} = 2079, \; c_{62} = -9459.$$  

At the same time the computations used in deriving the results of section 4 yield

$$u_{1y}(\tilde{y}, s) \approx s^{-3/2}(3\sigma)^{-1/2} \{ -2 - 2w^{-2} - \frac{25}{9} w^{-3} + ... \}$$

and

$$u_{2y}(\tilde{y}, s) \approx s^{-3/2}(3\sigma)^{-1/2} \{ - \frac{4}{3} w^{-1} - \frac{4}{3} w^{-2} - \frac{52}{9} w^{-3} + ... \}$$

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In the case where the two costs are combined, the evaluation of the main term \( Ks^{-1/2} \varphi(\alpha) \) require knowledge only of the optimal boundary because the boundary conditions express \( u \) and \( u_y \) in terms of the stopping risk along the optimal boundary. Suppose however that the problem were modified. For example suppose that for \( t > 2 \), a prescribed non-optimal boundary must be used. Then all of our expansions for \( t \to 0 \) would remain unaltered except that the evaluation of the coefficient of \( s^{-1/2} \varphi(\alpha) \) would depend either on \( u_{ly} \) along the boundary for \( t > 2 \) or alternatively on \( \int u dy \) where the integral corresponds to that part of the line \( t=2 \) which is in the continuation region (for \( t=2- \)). Thus we see that in many of these modified problems there is no essential difference in the problem of finding the coefficient of \( s^{-1/2} \varphi(\alpha) \) for the combined risk and for the separate risks.
10. Operating Characteristics.

In cases where the a priori distribution is not necessarily normal or where the statistician is not willing to accept an a priori probability distribution at all, it becomes important to derive the operating characteristics of the procedure studied. Each starting point \((x,t)\) characterizes a procedure. For each starting point and each value of the mean \(\mu\), the functions \(T(\mu; x,t)\), \(\beta(\mu; x,t)\), and \(R(\mu; x,t)\) = \(T + |\mu|\beta\) represent the expected sampling cost, the probability of error, and the risk respectively. Expressing these functions in terms of \(v = x - \mu t\) and \(t^* = t\), they satisfy the diffusion equations

\[
1 + \frac{1}{2} T_{vv} + T_t = 0
\]

\[
\frac{1}{2} \beta_{vv} + \beta_t = 0
\]

and

\[
1 + \frac{1}{2} R_{vv} + R_t = 0
\]

with the obvious boundary conditions. While the techniques of this paper should suffice to yield asymptotic expansions for \(t \to 0\), there may be some difficulty in establishing that these expansions give approximations which hold uniformly in \(x\) or \(\alpha\).

It seems reasonable to expect that as \(t \to 0\) for \(\alpha\) bounded, \(T\), \(\beta\), and \(R\) converge to functions of \(\mu\) which are independent of \(\alpha\). When \(\mu\) has the normal a priori distribution \(\mathcal{N}(\alpha_o^{-1/2}, t_o^{-1})\), \(t_o\) small, the density of \(\mu\) is approximately \(t_o^{1/2} \varphi(\alpha_o)\) over a large range of \(\mu\). Then the asymptotic behavior \(B(x,t) \approx C_o t^{1/2} \varphi(\alpha)\) is consistent with the above mentioned expectation and the further one that \(\int_{-\infty}^{\infty} R(\mu; x,t) d\mu \approx C_o = \int_{-\infty}^{\infty} R(\mu; 0,0) d\mu\).
This conjecture may be applied to the non-normalized problem as follows. Suppose that the cost per unit time of observation (in the t* scale) is c* (small) the cost of a wrong decision is k*|μ*|, μ* is the mean drift per unit time of the observed Wiener process which has variance σ*² per unit time. Let the not necessarily normal a priori distribution of μ* be G* which has a density approximately g*(o) in the neighborhood of μ* = 0. Then according to the normalizing transformations of [2, §3],

$$R* = c*^{1/3}k*^{2/3}o*^{2/3}$$

where R* is the Bayes risk for the normalized problem involving

$$μ = c*^{-1/3}k*^{1/3}o*^{1/3}μ*$$

whose a priori distribution G* has a density which is approximately g*(o)c*^{1/3}k*^{-1/3}o*^{2/3} over a large range of μ. Then one would expect

$$R ≈ \int R(μ; o, o)dG(μ) ≈ c*G*(o)c*^{1/3}k*^{-1/3}o*^{2/3}$$

and

$$(10.1) \quad R* ≈ c*G*(o)c*^{2/3}k*^{1/3}o*^{4/3}.$$ 

To put some rigor into a derivation of (10.1) it would be necessary to develop some bounds of R(μ; o, t) or R(μ; o, o) as μ becomes large. We terminate this section with a brief sketch of such a derivation. A tool would be the distribution of the time at which a Wiener process Z_t with mean 0 and variance one per unit time first crosses a straight line a+mt. Implicit in Wald [8, p. 191-3] is the result that the distribution has density
\[ f_{\alpha, \mu}(t) = (2\pi)^{-1/2} \alpha^{-3/2} \exp\left(-\frac{1}{2} \left[ \alpha^{-1/2} + \mu t \right]^2 \right), \quad t > 0, \alpha > 0, \]

and characteristic function \( \exp[a(m - \sqrt{m^2 - 2\lambda})] \). It should be noted that when \( m > 0 \), the probability of crossing the line is less than one but the above formulae are still correct and yield a total probability of crossing the line equal to \( \exp(-2\alpha m) \).

The optimal boundary may be approximated from above by \( \tilde{x}^+ \) and from below by \( \tilde{x}^- \) where both of these approximations are concave for \( t \) sufficiently small. Consider a large specified value of \( \mu \). The process \( X_t \) will intersect the boundary at least as soon as it will intersect a line which is tangent to \( \tilde{x}^+ \) for some small \( t \). Take the tangent line at the point where \( \mu t = \tilde{x}^+(t) \). The process \( X_t - \mu t \) is a Wiener process with zero drift and the values of \( m \) and \( \alpha \) corresponding to the above tangent line are \( \frac{dx^+}{dt} \mu < 0 \) and \( \tilde{x}^+ - t \frac{dx^+}{dt} \). From the characteristic function the mean time till \( X_t \) intersects the tangent line is \( a/m \). Substituting \( \tilde{x}^+ = \sqrt{t(-3 \log t - \log 8\pi + 2(\log t)^{-1}} \ldots \) we have \( a/m \approx t \approx 3 \log \mu^2/\mu^2 \). Hence for large \( \mu \),

\[ T(\mu; \alpha, \lambda) \leq 3\mu^{-2}(\log \mu^2)(1+o(1)) \]

Because the above bound is approximately the time required for the "mean drift line" to intersect the boundary, it isn't difficult to show that it is a good approximation to \( T \).

The probability of error is dominated by the probability that the process crosses \( \tilde{x}^-(t) \). The probability that it first crosses this boundary between \( t \) and \( t+dt \) (\( t \) small) is less than the probability
that it first crosses the tangent line to \( \tilde{x}(-t) \) in this time interval. Hence small values of \( t \) contribute at most

\[
\int (2\pi)^{-1/2} a(t)t^{-3/2} \exp\left\{ -\frac{1}{2}[a(t)t^{-1/2} + m(t)t^{-1/2}]^2 \right\} dt
\]

to the error. Substituting \( a(t) = \tilde{x}^-_t \frac{d\tilde{x}^-}{dt} \) and \( m(t) = \frac{d\tilde{x}^-}{dt} + \mu > 0 \), observing where \( a(t)t^{-1/2} + m(t)t^{-1/2} \) is minimized, and applying the steepest descent argument, the above integral is approximately \( O(\mu^{-3}(\log \mu^2)^{-1}) \).

The section where \( t \) is not small can be disposed of easily. Thus we have

\[
(10.4) \quad \beta(\mu; 0, 0) = O(\mu^{-3}(\log \mu^2)^{-1}) .
\]

Thus, the contribution of error to risk is smaller than that of sampling by a factor of \( (\log \mu^2)^2 \) when \( \mu \) is large.
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