LARGE-SAMPLE ESTIMATION OF AN UNKNOWN DISCRETE WAVEFORM WHICH IS RANDOMLY REPEATING IN GAUSSIAN NOISE

BY
MELVIN HINICH

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DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
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LARGE-SAMPLE ESTIMATION OF AN UNKNOWN DISCRETE WAVEFORM
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I. Summary and Introduction.

Suppose we have an input \( X(t) \) made up of an unknown waveform \( \theta(t) \) of known length, which is repeated randomly, and is imbedded in Gaussian noise with a known covariance function. The rate of recurrence of the waveform is a known small constant. In addition, the signal-to-noise ratio of the input \( X(t) \) is quite low. We wish to estimate the waveform \( \theta(t) \) and its autocorrelation \( \psi(\tau) = \int \theta(t+\tau)\theta(t)dt \). Restricting ourselves to discrete-time observations on \( X(t) \), we shall derive an optimal estimator of the discrete version of \( \psi(\tau) \). This estimator is a weighted average of the sample autocorrelation and the square of a linear estimator of the time average (the zero-frequency of DC value) of the waveform. For the estimation of \( \theta \), the problem is more complicated. The optimality concept (asymptotic efficiency) used in this work is based upon large-sample theory and the Cramér-Rao Inequality (Chernoff [2] and Cramér [4]).

The problem stated above was motivated by a problem of electronic surveillance of an enemy communication system based upon pulse position modulation, PPM. To illustrate this system suppose station A is sending a message, which we wish to intercept and decode, to station B by PPM over a certain FM bandwidth. A continues to repeat a fixed pulse-type waveform \( \theta(t) = \sum_{i=1}^{N} \theta_i H[t-(i-1)T] \) where
\[ H(t) = \begin{cases} 
1 & \text{if } 0 \leq t < T \\
0 & \text{otherwise.} 
\end{cases} \]

The vector \( \theta' = (\theta_1, \ldots, \theta_n) \), the parameter \( n \), and \( T \) - the pulse width - are known to both \( A \) and \( B \). Notice that \( nT \) is the time duration (length) of \( \theta(t) \).

Since \( \theta \), \( n \), and \( T \) are known to the receiver \( B \), one may ask what the coding scheme is for the information that \( A \) is sending. The answer is that the length of time between successive recurrences of the waveform, is the variable which contains the information.

In many applications the average length of time between successive occurrences of \( \theta(t) \) is around \( 10^2 \) times \( nT \). While the intervals between repetitions are fundamental in the transmission of information, someone who does not yet know the waveform may regard the recurrences to be purely random in time.

This modulation technique has the effect of spreading the power in the FM bandwidth over a wider swath of the frequency scale. This makes surveillance more difficult because it requires that we somehow determine the actual bandwidth being used. Moreover, the spreading of the power makes jamming of the channel difficult. To further complicate matters, \( A \) transmits the pulses with low power so that \( B \) picks up an input \( X(t) \) with low signal-to-noise ratio.

Since \( B \) knows \( \theta(t) \), he uses matched filtering to detect the times of occurrence of the \( \theta' \)'s. If the noise is assumed to be additive Gaussian noise with known covariance, then matched filtering is optimal in a decision theoretic sense (Wainstein and Zubakov [8]).
But suppose that we are listening in on this channel without knowing \( \theta \) and we wish to find out what \( A \) is saying. First we must determine the frequency band of the channel which \( A \) is using. Then we must detect the times of occurrence of the \( \theta \)'s, although we do not know \( \theta \). However, let us assume that we have already determined \( n \) and \( T \), although it will turn out that \( n \) is not a vital parameter in the estimator developed in this paper.

Jakowitz, Shuey, and White [6] present a special discrete-time (sampled data) system, called the Adaptive Filter, which estimates an unknown waveform which is repeating in additive noise. The system uses a complicated stochastic iterative procedure. The Filter obtains a crude estimate of the waveform from the initial input and uses the discrete cross-correlation between this estimate and the input to detect the times of occurrence of the waveform. When it decides that a waveform is present in the input, it refines the estimate by averaging it with the section of input where the waveform is thought to be present. Provided the autocorrelation of the waveform \( \Psi(\tau) \) has \( \Psi(0) \), its maximum, a good deal larger than the relative maxima of \( \Psi \), and provided the noise is well behaved, then this iterative procedure results in an asymptotically stable estimate for the waveform. This stable estimate is then used as the matching element in matched filter detection of the waveform. Thus we could call the Adaptive Filter an adaptive matched filter. A partial analysis of this system is given by Hinich [5].

The estimate of the discrete autocorrelation of the waveform is helpful to the analysis of systems which are based upon discrete-time
cross-correlation such as the Adaptive Filter, since the autocorrelation is a basic parameter in the distributions of the random variables (correlations) which arise in the operation of these systems.

Suppose we can obtain an expression for $\theta$ in terms of $\psi$, $\theta = f(\psi)$. Once we have obtained an asymptotically efficient estimator of $\psi$, call it $\hat{\psi}$, then $\hat{\theta} = f(\hat{\psi})$ would be an asymptotically efficient estimator of $\theta$. Unfortunately there is a multiplicity of $\theta$'s which have $\psi$ as their autocorrelation. In Section VI we will present a method for obtaining the correct $\theta$ (the one which appears in the input) from $\psi$ by using the observations on $X(t)$. Unfortunately this method does not seem to be sufficiently practical.

To conclude, let us outline the rest of this paper. In Section II we give a formal statement and description of the problem posed above. In Section III we state the Cramér-Rao Theorem and derive the information matrices relevant to the estimation of $\theta$ and $\psi$, as well as their inverses. In Section IV we present the optimal estimator of $\psi$. We also show that while the normalized sample correlation is an unbiased estimator of $\psi$, it is not efficient. In Section V we discuss three examples. These are the general case white noise, the case of white noise when the discrete waveform has only two components $\theta_1, \theta_2$, and the case of Gaussian Markov noise ($E N(t+\tau)N(t) = \rho |\tau|$, $0 < \rho < 1$). In Section VI we discuss the problem of estimating $\theta$ after the autocorrelation $\psi$ has been estimated.
II. Statement of Problem.

We shall develop a formal statement of the problem. First let us discuss it informally.

We observe a process $X(t)$ which consists of a randomly occurring unknown waveform plus noise. The noise process $N(t)$ is assumed to be stationary and Gaussian with mean zero and known covariance. Without loss of generality we may normalize so that the noise has variance $\sigma^2_N(t) = EM^2(t) = 1$.

The waveform $\theta(t)$ has known specified length and can be represented as a step function as follows:

$$\theta(t) = \sum_{i=1}^{n} \theta_i H[t-(i-1)T], \quad n,T \text{ known},$$

$$H(t) = \begin{cases} 
1 & \text{ if } 0 \leq t < T \\
0 & \text{ otherwise.}
\end{cases}$$

The parameter $T$ represents the width of the steps of the step function, i.e., the pulse width. The number of equally spaced steps corresponding to the waveform $\theta(t)$ is given by $n$.

We assume that the time intervals between repetitions are large compared to the length of the waveform. Let $\gamma$ be the rate of repetitions measured in units of $T$ seconds (pulse width). Then $\gamma^{-1}T$ seconds is the average time between waveforms, and thus, $\gamma$ is small.

The ultimate objective is to estimate $\theta' = (\theta_1, \ldots, \theta_n)$. However, the discrete autocorrelation $\psi' = (\psi_1, \ldots, \psi_n)$ defined by:
\[
\psi_1 = \frac{1}{2} (\theta_1^2 + \theta_2^2 + \cdots + \theta_n^2) \\
\psi_2 = \theta_1 \theta_2 + \theta_2 \theta_3 + \cdots + \theta_{n-1} \theta_n \\
\psi_3 = \theta_1 \theta_3 + \theta_2 \theta_4 + \cdots + \theta_{n-2} \theta_n \\
\vdots \\
\psi_n = \theta_1 \theta_n
\]

is an especially useful function of \( \theta \). Also useful is the discrete
time-average

\[
\psi_0 = \sum_{i=1}^{n} \theta_i \quad \text{(DC value of } \theta) \nonumber.
\]

We assume that \( \| \theta \| = (\sum_{i=1}^{n} \theta_i^2)^{\frac{1}{2}} \) is small compared to the variance
of the noise. We can state this in terms of \( R_{\theta} \), the signal-to-noise
ratio of \( X(t) \). By definition \( R_{\theta} = \frac{1}{nT} \sum_{i=1}^{n} \theta_i^2 / \int_{-\infty}^{\infty} S_N(f) \, df \) where
\( S_N(f) \) is the spectral density of the noise \( N(t) \). But \( \int_{-\infty}^{\infty} S_N(f) \, df = \text{E}(N^2)(t) = \sigma_N^2(t) = 1 \) by the normalization of \( N \). Thus, \( R_{\theta} = \frac{1}{nT} \sum_{i=1}^{n} \theta_i^2 = \frac{\| \theta \|^2}{nT} \) and we assume that \( R_{\theta} \) is small.

Now we will discuss the sampling procedure. For large \( m \) and a
fixed integer \( w \), we will take \( m \) groups of \( w \) successive discrete
observations on \( X(t) \); the successive observations being \( T \) seconds
apart. That is for each \( t_i \) such that \( t_1 < t_2 < \cdots < t_m \), we
observe

\[
X(t_i + T), X(t_i + 2T), \ldots, X(t_i + wT) .
\]
The intervals between the \( t_i \)'s \((t_{i+1} - t_i)\) are all substantially greater than \( wT \) and \( nT \) (the time duration of the waveform). Let

\[
X^{(i)} = (X(t_i + T), \ldots, X(t_i + wT)) \quad i = 1, \ldots, m.
\]

We thus have a sample of \( m \) vector observations on a \( w \)-dimensional random variable.

This sampling scheme may be regarded as opening a sequence of windows of width \( wT \) seconds, through which we observe the process at \( m \) different stages of time.

There are several different possibilities which can occur when a window is opened. There may be no part of the waveform present during the \( wT \) second "look" at \( X(t) \). In that case we observe only noise. However, the window may open just as the front part of the waveform is "visible". In that case only the head of the vector \( \theta \) plus noise is observed. Similarly, we might observe only the tail of \( \theta \) plus noise, or perhaps the middle of \( \theta \) plus noise. Incidentally let us suppose that \( wT < \gamma^{-1} T \) since we wish to exclude the possibility of catching two successive waveforms in the window.

Since the window has \( w \) components and \( \theta \) has \( n \) components, there are \( n+w-1 \) ways of catching part of \( \theta \) along with the noise. We can represent these \( n+w-1 \) possibilities for \( \theta \) in the window by defining

\[
(3) \quad (S_j \theta)' = (\theta_{j+1}', \ldots, \theta_{j+w}')
\]

where \( \theta_k = 0 \) if \( k \leq 0 \) or \( k \geq n+1 \).
For example \((S_{n-1} \theta)' = (\theta_n, 0, \ldots, 0)\) and \((S_{-w+1} \theta)' = (0, 0, \ldots, \theta_1)\). There is no way of knowing in advance which case is occurring. Each of the \(n+w-1\) possibilities may be regarded as equally likely. Figure 1 gives an example for \(w=3, n=2\) where the noise has been removed.

Figure 1. Example of Sampling System with Noise Removed and \(n=2, w=3\).

We will catch some part of \(\theta\) in the \(i^{th}\) window if and only if \(\theta(t)\) commences at time \(t_o: -nT + t_i + T < t_o < t_i + wT\). Thus, the probability of this event is approximately \((n+w-1)\gamma\) where \(\gamma\) is the recurrence rate. The probability for observing a specific one of the \(n+w-1\) possibilities is simply \(\gamma\).

Since the distances between windows are greater than \(nT\), a single waveform cannot appear in two successive windows. Moreover, suppose that for some \(\tau_o, \mathbb{E}N(t+\tau)N(t) = 0\) for \(\tau > \tau_o\). Then if we take the windows further apart than \(\tau_o\) seconds, the \(X(i)\)'s are independent. The above restraint on the covariance of the noise holds
approximately for many colored noise processes which occur in applications. Of course, for white noise, \( \tau_c = 0 \).

We then can sum up this discussion with a formal statistical statement of an idealized version of the problem:

We have \( m \) independent \( w \)-dimensional random vectors \( X^{(1)}, X^{(2)}, \ldots, X^{(m)} \), each identically distributed as \( X \) where

\[
X = \begin{cases} 
N + S_{-w+1}^\theta & \text{with probability } \gamma \\
N + S_{-w+2}^\theta & \text{" " "} \\
\vdots & \text{" " "} \\
N + S_{n-2}^\theta & \text{" " "} \\
N + S_{n-1}^\theta & \text{" " "} \\
N & \text{" " 1-(n+w-1)\gamma}
\end{cases}
\]

and \( S_i^\theta \) is defined in (3).

The vector random variable \( N \) has a \( w \)-dimensional multivariate normal distribution with mean zero and known, non-singular covariance matrix \( \Sigma \). We express this by \( \mathcal{L}(N) = \mathcal{N}(0, \Sigma) \) where \( \mathcal{L}(N) \) is the distribution function of the random variable \( N \).

We desire to estimate the \( n \)-dimensional vectors \( \theta \) and \( \psi \)

\[
(\psi_1 = \frac{1}{2} \sum_{i=1}^{n} \theta_i^2, \psi_2 = \sum_{i=1}^{n-1} \theta_i \theta_{i+1}, \ldots, \psi_n = \theta_1 \theta_n) \text{ where it is assumed that } \gamma \text{ is small, } ||\theta|| = \left[ \sum_{i=1}^{n} \theta_i^2 \right]^{1/2} \text{ is small and the sample size } m \text{ is large.}
\]
III. Information Matrix.

The Cramér-Rao Theorem gives a bound for the asymptotic variance of estimators in terms of the inverse of the information matrix (defined below). In this section we will compute the information matrix and its inverse for the given problem.

Given a parameter vector \( \varphi' = (\varphi_1, \ldots, \varphi_n) \) and a random variable \( X \) with density function \( p(x|\varphi) \), the information matrix for \( p \) and \( \varphi \), \( I(\varphi) \) is defined by:

\[
I(\varphi) = E_{\varphi} \left[ \frac{\partial \log p(x|\varphi)}{\partial \varphi} \right] \left[ \frac{\partial \log p(x|\varphi)}{\partial \varphi} \right]' : n \times n
\]

where

\[
\left[ \frac{\partial \log p(x|\varphi)}{\partial \varphi} \right]' = \left( \frac{\partial \log p(x|\varphi)}{\partial \varphi_1}, \ldots, \frac{\partial \log p(x|\varphi)}{\partial \varphi_n} \right)
\]

and \( E_{\varphi} \) represents expectation with respect to \( p(x|\varphi) \).

Consider a random vector \( T(X)' = (T_1(X), \ldots, T_n(X)) \). \( T(X) \) is called an unbiased estimator of \( \varphi \) if \( E_{\varphi} T(X) = \varphi \). Furthermore we call

\[
K_T = E_{\varphi} [T(X) - E_{\varphi} T(X)][T(X) - E_{\varphi} T(X)]'
\]

the covariance of \( T \), thus if \( T \) is an unbiased estimator of \( \varphi \),

\[
K_T = E_{\varphi} [T(X)-\varphi][T(X)-\varphi]'.
\]

The Cramér-Rao Inequality states that if \( T \) is an unbiased estimator of \( \varphi \), \( K_T \geq I^{-1}(\varphi) \) where the inequality between these matrices means that for any vector \( a \), \( a'K_T a \geq a'I^{-1}(\varphi)a \).
But suppose we have a random sample \( X_1, \ldots, X_m \) of \( m \) independent observations from a population with density \( p(x|\varphi) \). Let \( \{T_m(X_1, \ldots, X_m): m=1,2,\ldots\} \) be a sequence of estimators of \( \varphi \) such that

\[
\sqrt{m} (T_m - \varphi) \to \mathcal{N}(0,K) \quad \text{as} \quad m \to \infty.
\]

That is the distribution functions of \( \sqrt{m} [T_m(X_1, \ldots, X_m)] \) converge to an \( n \)-dimensional multivariate normal distribution with covariance matrix \( K \). Then \( K \) is called the asymptotic covariance matrix of \( \{T_m\} \).

Paraphrasing Stein's generalization of the Cramér-Rao Theorem (see Theorem 1 in Chernoff [2]), there does not exist any estimator of \( \varphi \) which has an asymptotic covariance matrix \( K \) such that for some vector \( a \), \( a'Ka < a'I^{-1}(\varphi)a \), for all \( \varphi \) in some open set \( S \).

Thus \( I^{-1}(\varphi) \) is "essentially" the lower bound for the asymptotic covariance of estimators of \( \varphi \). Thus we shall call the sequence \( \{T_m\} \) asymptotically efficient if \( K = I^{-1}(\varphi) \). Cramér [4] shows that under mild conditions, the maximum likelihood estimator is asymptotically efficient.

We will now deal with the information matrices of interest in this work. However, to facilitate the algebra we make the one-to-one transformation,

\[
Z = \sum^{-1} X
\]

where \( \sum^{-1} \) is the inverse of \( \sum \), the covariance matrix of the Gaussian noise vector \( N \). From (4) we have
\[
Z = \begin{cases} 
N^* + \sum_{j=-w+1}^{n-1} S_j \theta & \text{with probability } \gamma \text{ for each } j = -w+1, \ldots, n-1 \\
N^* \text{ with probability } 1-(n+w-1)\gamma 
\end{cases}
\]

where \( \mathcal{N}(N^*) = \mathcal{N}(0, \sum^{-1}) \).

Let \( f(z|\theta) \) be the density of \( Z \), parameterized by the waveform vector \( \theta \). Therefore from (7) we have

\[
f(z|\theta) = [1-(n+w-1)\gamma]n(z|0, \sum^{-1}) + \gamma \sum_{j=-w+1}^{n-1} n(z| \sum^{-1} S_j \theta, \sum^{-1})
\]

where

\[
n(z|\phi, C) = (2\pi)^{-\frac{w}{2}} \frac{1}{|C|} e^{-\frac{1}{2} (z-\phi)'C^{-1}(z-\phi)}
\]

is a \( w \)-dimensional normal density with mean \( \phi \) and covariance matrix \( C \).

Notice that \( f(z|\theta) \) is a convex combination of multivariate normal densities, but it is not in general multivariate normal itself. We shall handle it by making Taylor series approximations with \( \theta \) in the neighborhood of zero.

Let \( I(\theta) \) be the information matrix for \( f(z|\theta) \) and \( \theta \). As in (5),

\[
I(\theta) = E_{\theta} \left[ \frac{\partial}{\partial \theta} \log f(z|\theta) \right] \left[ \frac{\partial}{\partial \theta} \log f(z|\theta) \right]'
\]

But we also wish to deal with the information matrix for the auto-correlation \( \psi \) given in (1). We see that \( \psi(\theta) \) is not a one-to-one transformation for all \( \theta \). Let \( J \) be the Jacobian matrix of the transformation, \( J_{ij} = \frac{\partial \psi}{\partial \theta_i} \) with \( i, j = 1, \ldots, n \). It is easy to see that
where we recall that \( \theta_k = 0 \) if \( k \leq 0 \) or \( k \geq n+1 \). It is not difficult to see that

\[
|J| = [\sum_{i=1}^{n} \theta_i] [\sum_{i=1}^{n} (-1)^i \theta_i] P_{n-2}(\theta)
\]

where \( P_{n-2}(\theta) \) is a homogeneous \( n-2 \) degree polynomial in the \( \theta_i \)'s.

Therefore the set of \( \theta \)'s such that \( |J| = 0 \) form surfaces which divide the \( n \)-dimensional Euclidean space into at most \( 2^n \) regions where \( J \) does not vanish. We will restrict ourselves to \( \theta \) vectors such that \( J \) does not vanish. Thus \( J^{-1} \) exists for these \( \theta \)'s.

Moreover given a \( \theta_o \) such that \( |J| \neq 0 \), there exists a neighborhood of \( \theta_o \) in which \( \psi(\theta) \) is one-to-one.

Now we will deal with the approximations of \( f(z|\theta) \). From (8) we have

\[
\frac{f(z|\theta)}{f(z|0)} = 1-(n+w-1)\gamma + \gamma \sum_{j=-w+1}^{n-1} \frac{n(z|\sum -1S_j \theta, \sum -1)}{n(z|0, \sum -1)}
\]

since \( f(z|0) = n(z|0, \sum -1) \). By expanding a Taylor's series about \( \varphi = 0 \) we have
\begin{align}
(14) \quad \frac{n(z \mid \sum^{-1} \varphi, \sum^{-1})}{n(z \mid 0, \sum^{-1})} &= 1 + z' \varphi + \frac{1}{z} \varphi'(zz' - \sum^{-1}) \varphi \\
&+ \sum_{i,j,k=1}^{W} G_{3}(z \mid i, j, k)c_{3}(i,j,k)\varphi_i \varphi_j \varphi_k \\
&+ \sum_{i,j,k,\ell=1}^{W} G_{4}(z \mid i, j, k, \ell)c_{4}(i,j,k,\ell)\varphi_i \varphi_j \varphi_k \varphi_\ell \\
&+ \|\varphi\|^5 K^*(z, \varphi)
\end{align}

where the c's are constants and

\[ |K^*(z, \varphi)| \leq d e^{-6 \sum_{i=1}^{t^*} |z_i| + \cdots + t^* \sum_{i=1}^{t^*} |z_i|} \]

for some \( t^*_i > 0 \) and \( d > 0 \). Thus \( E_{o}[K^*(z, \varphi)]^r \) exists and is bounded by some number independent of \( \varphi \) for each \( r \geq 0 \). With the notation

\[ \sum^{-1} = (\sigma^{ij}) \]

\begin{align}
(15a) \quad G_{3}(z \mid i, j, k) &= z_{i} z_{j} z_{k} - \sigma_{i j} z_{k} - \sigma_{i k} z_{j} - \sigma_{j k} z_{i} \\
(15b) \quad G_{4}(z \mid i, j, k, \ell) &= z_{i} z_{j} z_{k} z_{\ell} - 2\sigma_{i j} z_{k} z_{\ell} - 2\sigma_{i k} z_{j} z_{\ell} \\
&- 2\sigma_{i \ell} z_{j} z_{k} + \sigma_{i j} \sigma_{k \ell} + \sigma_{i k} \sigma_{j \ell} + \sigma_{i \ell} \sigma_{j k} \]
\end{align}

From page 39 of Anderson [1],

\[ E_{o} Z_{i} Z_{j} Z_{k} Z_{\ell} = \sigma_{i j} \sigma_{k \ell} + \sigma_{i k} \sigma_{j \ell} + \sigma_{i \ell} \sigma_{j k} \]

\[ E_{o} Z_{i} Z_{j} = \sigma_{i j} \]

Moreover all odd moments of the \( Z_1's \) are zero. We then have the following orthogonality relationships:
\[ E_{o}Z_{i}(Z^{j}_{j} - \sigma^{jk}) = 0 \quad \text{for all } i, j, k \]
\[ E_{o}Z_{i}G_{j}(Z^{j}_{j}, k, \ell) = 0 \quad \text{for all } i, j, k, \ell \]
\[ E_{o}Z_{i}G_{j}(Z^{j}_{j}, k, \ell, m) = 0 \quad \text{for all } i, j, k, \ell, m \]
\[ E_{o}(Z^{i}_{i}Z_{j} - \sigma^{ij})G_{j}(Z^{j}_{j}, k, \ell, m) = 0 \quad \text{for all } i, j, k, \ell, m. \]

Putting \( \varphi = S_{j} \theta \) in (14) and summing, we have from (13)

\[ \frac{\varphi(z, \theta)}{\varphi(z, 0)} = 1 + \gamma [z' \left( \sum_{j} S_{j} \theta \right) + \frac{1}{2} \sum_{j} (S_{j} \theta)' (zz' - \sum^{-1}) (S_{j} \theta)] + \sum_{i,j,k} G_{j}(z, i, j, k) o(\|\theta\|^3) \]
\[ + \sum_{i,j,k, \ell} G_{i}(z, i, j, k, \ell) o(\|\theta\|^4) \]
\[ + K(z, \theta) o(\|\theta\|^5) \]

\( E_{o}[K(z, \theta)]^r \) exists and is bounded by some number independent of \( \theta \) for each \( r \geq 0 \). Moreover the \( o(\|\theta\|^3) \) and \( o(\|\theta\|^4) \) terms are functions of \( \theta \) which do not involve \( z \).

Applying (1), (2), and (3) we have

\[ \sum_{j=-w+1}^{n-1} S_{j} \theta = \Psi_{0} \psi_{w} \]

where \( \psi_{w} = (1, \ldots, 1) \), and for any symmetric \( w \times w \) matrix \( A = (a_{ij}) \)

\[ \frac{1}{2} \sum_{j=-w+1}^{n-1} (S_{j} \theta)' A(S_{j} \theta) = \left( \sum_{i=1}^{w} a_{ii} \right) \psi_{1} + \left( \sum_{i=1}^{w-1} a_{i,i+1} \right) \psi_{2} \]
\[ + \cdots + \left( \sum_{i=1}^{w-n+1} a_{i,i+n-1} \right) \psi_{n}, \]

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where \( a_{ij} \) is understood to be zero when \( i \) or \( j \) is greater than \( w \) or less than \( 1 \). Thus, if \( w < n \), the coefficients of \( \psi_{w+1}, \ldots, \psi_n \) vanish and

\[
\frac{1}{2} \sum_j (S_j \theta)' A(S_j \theta) = \left( \sum_{i=1}^{w} a_{ii} \right) \psi_1 + \cdots + a_{1w} \psi_w.
\]

Applying (17) and (18) to (16) we have the following result.

**Lemma 1.** Define \( Y(z)' = (Y_1(z), \ldots, Y_n(z)) \) by

\[
Y_k(z) = \sum_{i=1}^{w-k+1} (z_i z_{i+k-1} - \theta_i, i+k-1), \quad k=1, \ldots, n.
\]

Then

\[
\frac{f(z|\theta)}{f(z|0)} = 1 + \gamma ([z'_{1w}] \psi_0 + Y(z)' \psi) + \sum_{i,j,k} G_{ji}(z|j,k) O(\|\theta\|^3)
\]

\[
+ \sum_{i,j,k} G_{ji}(z|j,k) O(\|\theta\|^4)
\]

\[
+ K(z, \theta) O(\|\theta\|^5).
\]

It is understood that \( z_i = 0 \) if \( i \leq 0 \) or \( i \geq w+1 \), and thus if \( w < n \)

\[
Y_{w+1}(z) = \ldots = Y_n(z) = 0.
\]

We then define for \( w < n \)

\[
Y(z|w)' = (Y_1(z), \ldots, Y_w(z)) .
\]

We now introduce the \( n \times n \) matrix.

\[
(21a) \quad D = E_0 Y(z) Y(z)' = (d_{ij}).
\]
If \( w < n \) we can write
\[
D = \begin{pmatrix} D_w & 0 \\ 0 & 0 \end{pmatrix}
\]

where
\[
D_w = E_0 Y(Z|w) Y(Z|w)'
\]
is a \( w \times w \) matrix. From (19) we can show for \( 1, j = 1, 2, \ldots, n \) that
\[
(21b) \quad d_{ij} = \sum_{k=1}^{w-1} \sum_{\ell=1}^{w-1} (d_{k\ell} d_{k+i-1, \ell+j-1} + d_{k, \ell+j-1} d_{k+i-1, \ell})
\]
where it is understood that \( d_{k\ell} \) is zero if \( k \) or \( \ell \) is greater than \( w \) or less than \( 1 \).

**Lemma 2.** For \( w \geq n \), \( D \) is positive definite. For \( w < n \), \( D_w \) is positive definite.

**Proof.** Let \( w \geq n \) and notice that \( E_0 Y(Z) = 0 \). Thus \( D \) is the covariance matrix of \( Y(Z) \) and is therefore non-negative definite.

Suppose \( D \) is singular. Then there exists a non-zero vector \( a \) such that \( a' Y(Z) = 0 \) with probability one, which implies that,
\[
2a_1 U_1 + a_2 U_2 + \ldots + a_n U_n = \text{constant}
\]
where
\[
U_1 = \frac{1}{2} \sum_{k=1}^{w} Z_k^2
\]
\[
U_2 = \sum_{k=1}^{w-1} Z_k Z_{k+1}
\]
\[
\vdots
\]
\[
U_n = \sum_{k=1}^{w-n+1} Z_k Z_{k+n-1}
\]
\[
\vdots
\]
\[
U_w = Z_1 Z_w
\]
We see that \( U(Z)' = (U_1', \ldots, U_n') \) results from an autocorrelation operation on the random vector \( Z \). Therefore the Jacobian matrix of the transformation \( U(Z) \) is given by (10) with \( n \) replaced by \( w \) and \( \theta_k \) by \( Z_k \).

As in the case of the transformation from \( \theta \) to \( \psi \), there exists a point \( z_0 \) in a neighborhood of which (i) the Jacobian is non-zero, and (ii) restricted to this neighborhood \( U \) is a one-to-one transformation mapping \( w \)-dimensional open sets onto \( w \)-dimensional open sets.

Now \( Z \) is a non-degenerate \( w \)-dimensional multivariate normal random variable and every region in the \( Z \) space has positive probability. Hence there is a neighborhood in the \( U \) space such that every \( w \)-dimensional cube in that neighborhood has positive probability. But then it is impossible that \( U \) be on the hyperplane

\[
2a_1 U_1 + a_2 U_2 + \cdots + a_n U_n = \text{constant}
\]

with probability one. Hence \( D \) is non-singular.

For \( w < n \), a similar argument holds for the hyperplane

\[
2a_1 U_1 + a_2 U_2 + \cdots + a_w U_w = \text{constant}.
\]

We will now approximate \( I(\theta) \). From (5) we rearrange terms and write,

\[
I(\theta) = A(\theta) - B(\theta) + C(\theta)
\]

where

\[
A(\theta) = E \left[ \frac{1}{f(Z|\theta)} \frac{\partial f(Z|\theta)}{\partial \theta} \right] \left[ \frac{1}{f(Z|\theta)} \frac{\partial f(Z|\theta)}{\partial \theta} \right]'
\]

\[
B(\theta) = E \left[ \frac{1}{f(Z|\theta)} \frac{\partial f(Z|\theta)}{\partial \theta} \right] \left[ \frac{1}{f(Z|\theta)} \frac{\partial f(Z|\theta)}{\partial \theta} \right]' \left[ \frac{f(Z|\theta)}{f(Z|\theta)} e^{-1} \right]
\]

\[
C(\theta) = E \left[ \frac{1}{f(Z|\theta)} \frac{\partial f(Z|\theta)}{\partial \theta} \right] \left[ \frac{1}{f(Z|\theta)} \frac{\partial f(Z|\theta)}{\partial \theta} \right]' \left[ \frac{f(Z|\theta)}{f(Z|\theta)} - 1 \right]
\]
\[ C(\theta) = E_o \left\{ \frac{1}{f(Z|0)} \frac{\partial f(Z|\theta)}{\partial \theta} \right\} \left[ \frac{1}{f(Z|0)} \frac{\partial f(Z|\theta)}{\partial \theta} \right]' \left[ \frac{f(Z|\theta)}{f(Z|0)} - 1 \right]^2 \frac{f(Z|0)}{f(Z|\theta)} \right\} \]

and

\[ \left[ \frac{\partial f(Z|\theta)}{\partial \theta} \right]' = \left\{ \frac{\partial f}{\partial \theta_1}, \ldots, \frac{\partial f}{\partial \theta_n} \right\} . \]

The Taylor expansion argument which gave (20) in Lemma 1 also yields

\[(24) \quad \frac{1}{f(Z|0)} \frac{\partial f(Z|\theta)}{\partial \theta} = \gamma [(1'w)l_n + JY(z) \]

+ \[ \sum_{i,j,k} G_3(z|i,j,k)O(\|\theta\|^2) \]

+ \[ \sum_{i,j,k,\ell} G_4(z|i,j,k,\ell)O(\|\theta\|^3) \]

+ \[ K(z,\theta)O(\|\theta\|^4) \]

where \( J = \frac{\partial \psi}{\partial \theta} \) is the Jacobian given by (10). Using (24) in (23) and noting that \[ \frac{f(z|0)}{f(z|\theta)} \leq \frac{1}{1-(n+w-1)\gamma} \], we have

\[(25a) \quad A(\theta) = \gamma^2 \left\{ (1'w) \sum_{-1}^{1} l_{w} l_n l_n + J\psi' + O(\|\theta\|^4) \right\} , \]

\[ B(\theta) = \gamma^3 \left\{ [E_o(1'Z)^2Y(Z)]'\psi(l_n l_n) \right\} \]

+ \[ \psi_o [JE_o(1'Z)^2Y(Z)]_n + \sum [E_o(1'Z)^2Y(Z)]_n \]

+ \[ O(\|\theta\|^2) \] ,

\[(25b) \quad B(\theta) = \gamma^2 O(\|\theta\|^2) , \]

\[(25c) \quad C(\theta) = \gamma^4 O(\|\theta\|^2) \]
since the G's and K have bounded moments with respect to n(z|0, Σ⁻¹).

Hence from (22)

(26) \[ I(θ) = γ^2 \left[ (L_n' \sum w_{i-1} l_n') J_n' + JDJ' \right. \\
\left. + O(γ\|θ\|^2) + O(\|θ\|^4) \right]. \]

Applying the chain rule (of differentiation with respect to θ) to

\[ ψ_o = \left(2 ∑_{i=1}^n ψ_i\right)^{1/2} = ∑_{i=1}^n θ_i \]

we have \[ ψ_o⁻¹ J_n = l_n \] and consequently

\[ J_n⁻¹ = ψ_o⁻¹ l_n. \]

Combining this with (26) we have

Lemma 3.

\[ I(θ) = γ^2 J \left[ \frac{l_n'}{ψ_o^2} \sum w_{i-1} l_n' \right. \\
\left. + D + O(γ) + O(\|θ\|^2) \right] J'. \]

If \( I(θ) \) is non-singular, then \( I⁻¹(θ) \) is the lower bound for the asymptotic covariance of estimators of θ. To calculate \( I⁻¹(θ) \) we shall make use of the following lemma which we state without proof.

Lemma 4. Let \( M = θ⁻¹ l l' + R \) where R is positive definite. Then,

\[ M⁻¹ = R⁻¹ - \frac{R⁻¹ l l' R⁻¹}{θ⁻1 R⁻¹ l} \]
\[ = R⁻¹ - \frac{R⁻¹ l l' R⁻¹}{l R⁻¹ l} + \frac{R⁻¹ l l' R⁻¹}{l R⁻¹ l} \left( R⁻¹ l l' \right) \]
\[ \alpha + O(α^2) \]
as \( α \to 0. \)

Now let \( ∑ (ψ,ψ_o) \) denote the Cramer-Rao lower bound for the asymptotic covariance of estimators of the n+1 parameters \( ψ_1, ψ_2, ..., ψ_n, ψ_o. \)
In order to obtain \( \sum (\psi, \psi_0) \) from \( I^{-1}(\theta) \), let us now state a result which was originally derived by a procedure called the delta method, (Mann and Wald [7]).

Let \( \{ T_m \} \) be a sequence of estimators of \( \varphi^* = (\varphi_1^*, \ldots, \varphi_R^*) \) such that

\[
\mathcal{L}(\sqrt{m}(T_m - \varphi^*)) \rightarrow \mathcal{N}(0, K) \text{ as } m \rightarrow \infty
\]

Let \( g \) be a vector-valued function of \( \varphi \) continuously differentiable at \( \varphi^* \). Then

\[
(27) \quad \mathcal{L}(\sqrt{m} \cdot g'(T_m - g(\varphi^*))) \rightarrow \mathcal{N}(0, [\frac{\partial g}{\partial \varphi}]' K [\frac{\partial g}{\partial \varphi}])
\]

where \( [\frac{\partial g}{\partial \varphi}] \) is a matrix whose \((i,j)\)th element is given by \( \frac{\partial g_i}{\partial \varphi_j} \). \( \varphi = \varphi^* \).

If we let \( \sum g \) denote the lower bound for the asymptotic covariance of estimators of \( g(\theta) \), then if \( I(\varphi) \) is non-singular

\[
(28) \quad \sum g = [\frac{\partial g}{\partial \varphi}]' I^{-1}(\varphi) [\frac{\partial g}{\partial \varphi}].
\]

We then have from (28) if \( I(\theta) \) is non-singular,

\[
(29) \quad \sum (\psi, \psi_0)' = \left[ -\frac{\partial (\psi, \psi_0)}{\partial \theta} \right]' I^{-1}(\theta) \left[ -\frac{\partial (\psi, \psi_0)}{\partial \theta} \right]
\]

where \( \frac{\partial (\psi, \psi_0)}{\partial \theta} \) is the Jacobian matrix with elements \( \frac{\partial \psi_j}{\partial \theta_i} \) for \( j=1,2,\ldots,n,0 \) and \( i=1,\ldots,n \).

Thus we can write

\[
(30) \quad \frac{\partial (\psi, \psi_0)}{\partial \theta} = (\mathcal{J}, \mathcal{J}_n)
\]
Applying (29), (30), and Lemmas 2, 3, and 4 we have

**Theorem 1.** If \( w \geq n \), then

\[
\sum (\psi, \psi_0) = \gamma^{-2} \left[ \begin{pmatrix}
D^{-1} - D^{-1} l'_n l^{-1} n & \psi_0 D^{-1} l_n \\
l'_n D^{-1} l_n & l'_w \sum_{-1}^{-1} l'_w l^{-1} n
\end{pmatrix}
\right] [I + O(\gamma) + O(\|\theta\|^2)]
\]

where \( I \) is the identity matrix.

**Corollary 1.1.** Let \( \sigma_{\psi_0}^2 \) denote the Cramér-Rao lower bound for the asymptotic variance of estimators of \( \psi_0 \). Then

\[
\sigma_{\psi_0}^2 = \gamma^{-2} [(l'_n \sum_{-1}^{-1} l_n)^{-1} + O(\gamma) + O(\|\theta\|^2)].
\]

**Corollary 1.2.** Let \( \sum \psi \) denote the Cramér-Rao lower bound for the asymptotic covariance of estimators of \( \psi \). Then

\[
\sum \psi = \gamma^{-2} [D^{-1} - D^{-1} l'_n l^{-1} n + O(\gamma) + O(\|\theta\|^2)].
\]

If \( w < n \) the main part of \( I(\theta) \), as represented in Lemma 3, is a singular matrix. From (20) we see that for \( \theta \) in the neighborhood of zero, \( f(z|\theta) \) has \( (\psi_1, \ldots, \psi_w) \) as natural parameters. We will obtain the lower bound for the asymptotic covariance of estimators of the \( n \) parameters \( \psi_1, \ldots, \psi_{n-1}, \psi_0 \), which will then give as a corollary the lower bound for the asymptotic covariance of estimators of \( \psi_0, \psi_1, \ldots, \psi_w \).
Let $\tau(\theta)' = (\psi_1', \ldots, \psi_{n-1}', \psi_o')$. Then the Jacobian matrix for $\tau(\theta)$ is,

$$
\frac{\partial \tau}{\partial \theta} = J \begin{pmatrix}
1 & 0 & \ldots & 0 & \psi_o^{-1} \\
0 & 1 & \ldots & 0 & \psi_o^{-1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & \psi_o^{-1} \\
0 & 0 & \ldots & 0 & \psi_o^{-1}
\end{pmatrix}
$$

It is assumed that $\theta$ is chosen such that $J$ is non-singular which implies that $\psi_o = \sum \theta_i \neq 0$. It then follows that $\frac{\partial \tau}{\partial \theta}$ is finite and non-singular and thus $\tau(\theta)$ is one-to-one in a neighborhood of $\theta$.

Let $f^*(z|\tau)$ be the density of $Z$ parameterized by $\tau$, $f^*(z|\tau) = f(z|\theta(\tau))$. Let $I^*(\tau)$ be the information matrix with respect to $f^*$ and $\tau$. Applying the chain rule we have,

$$
I^*(\tau) = (\frac{\partial \theta}{\partial \tau})^T I(\theta)(\frac{\partial \theta}{\partial \tau})'
$$

where $(\frac{\partial \theta}{\partial \tau}) = (\frac{\partial \tau}{\partial \theta})^{-1}$.

If $v < n$, $I^*$ is singular. Then define, the pseudo-inverse with respect to $P$ by

$$
I^*_{P^{-1}}(\tau) = \lim_{\lambda \to 0^+} (I^*(\tau) + \lambda P)^{-1}
$$

where $P$ is a symmetric matrix such that $I^* + \lambda P$ is positive definite for $\lambda > 0$. Chernoff [3] shows that the $(i,j)^{th}$ element of $I^*_{P^{-1}}$ is independent of the choice of $P$ if the $i^{th}$ and $j^{th}$ diagonal elements are finite. The submatrix of these invariant elements is the lower
bound for the asymptotic covariance for estimators of the $\Psi_i$ corresponding to these finite diagonal elements.

From (32) we can write

$$I_p^{*-1} = \lim_{\lambda \to 0^+} [(\frac{\partial^2}{\partial \psi^2})I(\theta)\frac{\partial \theta}{\partial \psi} + \lambda P]^{-1}$$

(33)

$$= \lim_{\lambda \to 0^+} (\frac{\partial^2}{\partial \theta^2})' [I(\theta) + \lambda Q]^{-1} (\frac{\partial^2}{\partial \theta^2})$$

where $Q = (\frac{\partial^2}{\partial \theta^2}) P (\frac{\partial^2}{\partial \theta^2})'$. If $w < n$, let

$$Q = J \begin{pmatrix} 0 & 0 \\ 0 & I_{n-w} \end{pmatrix} J'$$

(34)

where $I_{n-w}$ is the $(n-w) \times (n-w)$ identity matrix. From Lemma 3 and (34),

$$I(\theta) + \lambda Q = \gamma^2 \left[ J \left[ \sum_{\psi} \frac{1}{\psi^2} l_l l_l' \right] J' + D_W 0 \\ 0 \lambda I_{n-w} \right] + O(\gamma) + O(\|\theta\|^2)$$

(35)

We call the $r \times r$ upper left hand submatrix of the last 3 terms in the square brackets above $C_r$. Thus

$$C_r = \begin{pmatrix} D_W & 0 \\ 0 & \lambda I_{r-w} \end{pmatrix} + O(\gamma) + O(\|\theta\|^2), \ r = n-1, n$$

Applying (31), (33), (35) and lemmas 2 and 4, we have
$I_P^{-1} = \lim_{\lambda \to 0^+} \gamma^{-2} \left[ C^{-1} - \frac{C^{-1}_n l_n^{-1} C^{-1}_n l_n^{-1} C^{-1}_n l_n^{-1} C^{-1}_n l_n^{-1} C^{-1}_n l_n^{-1}}{l_n' C_n^{-1} l_n} \right] \left[ \psi_0 \frac{C^{-1}_n l_n^{-1}}{1_n' \sum_{l_n} l_n^{-1}} \frac{1_n' C_n^{-1} l_n^{-1}}{l_n' \sum_{l_n} l_n^{-1}} \right].$

\[ \left( I + O(\gamma) + O(\|\theta\|^2) \right) \]

From this it follows by use of matrix inequalities that

**Theorem 2.** If $w < n$

\[
I_P^{-1} \geq \gamma^{-2} \left[ \begin{array}{ccc}
D^{-1}_w & R_1 & 0 \\
R_1' & R_2 & R_3 \\
0 & R_3' (l_w' \sum_{l_w} l_w^{-1})^{-1} & 0
\end{array} \right] \left[ I + O(\gamma) + O(\|\theta\|^2) \right]
\]

where we leave the matrices $R_1$, $R_2$, and $R_3$ unspecified.

**Corollary 2.1.** If $w < n$

\[
\frac{\partial^2}{\partial_\psi_0} \geq \gamma^{-2} \left[ \frac{1}{l_w' \sum_{l_w} l_w^{-1}} + O(\gamma) + O(\|\theta\|^2) \right].
\]

**Corollary 2.2.** Let $\sum_{\psi(w)}$ be the Cramer-Rao lower bound for estimators of the $w$-dimensional vector $\psi(w)' = (\psi_1, \ldots, \psi_w)$. Then

\[
\sum_{\psi(w)} \geq \gamma^{-2} \left[ D^{-1}_w + O(\gamma) + O(\|\theta\|^2) \right].
\]

If $w = n - 1$, we can determine $\psi_n$ from $\tau$ because
\[
\psi_n - \frac{1}{2} \psi_0^2 - \sum_{j=1}^{n-1} \psi_j.
\]

Let \( \sigma^2_{\psi_n} \) be the Cramér-Rao bound for estimators of \( \psi_n \). Then

\[
\sigma^2_{\psi_n} = \left( \frac{\partial \psi_n}{\partial \tau} \right)^* \mathbf{I}_P^{-1} \left( \frac{\partial \psi_n}{\partial \tau} \right).
\]

Thus we have

**Corollary 2.3.** If \( w=n-1 \), \( \sigma^2_{\psi_n} \geq \gamma^{-2} \left[ \mathbf{I}_n^* \mathbf{D}_w^{-1} \mathbf{D}_n^{-1} + o(\gamma) + o(\|\theta\|^2) \right] \).
IV Estimation of $\psi$ and $\psi_0$.

The random vector $X(Z)$ defined in (19) as $X_k(Z) = \sum_{i=1}^{w-k+1} Z_i Z_{i+k-1} - \sigma^2_{1,i+k-1}$ could be called the autocorrelation statistic of $Z = \sum X$. An important example is the case where $\sum = I$, the identity matrix. Then

$$Y_1(X) = (X_1^2 - 1) + \cdots + (X_w^2 - 1)$$
$$Y_2(X) = X_1 X_2 + \cdots + X_w X_w$$
$$\vdots$$
$$Y_n(X) = X_1 X_n + \cdots + X_{w-n+1} X_w .$$

In this section we will give asymptotically efficient estimators of $\psi$ and $\psi_0$. These estimators are functions of $Y(Z)$ and $Z^1 w$.

In order to demonstrate efficiency, we will need the mean and covariance matrix of the $n+1$-dimensional random vector $(Y(Z)', Z^1 w)$.

**Lemma 5.**

$$E_\theta \left( \begin{array}{c} Y(Z) \\ Z^1 w \end{array} \right) = \gamma \left( \begin{array}{c} D \psi \\ 1 w \sum^{-1}_w \psi_0 \end{array} \right)$$

**Proof.** For any $\varphi$

$$\int (zz' - \sum^{-1} n(z | \sum^{-1} \varphi, \sum^{-1}) dz$$

$$= (\sum^{-1} \varphi)(\sum^{-1} \varphi)'$$

$$= \sum^{-1} \varphi \varphi', \sum^{-1} .$$

Thus from (7)

$$E_\theta (ZZ' - \sum^{-1}) = \gamma \sum^{-1} [ \sum_{j=w+1}^{n-1} (S_j \theta)(S_j \theta)' ] \sum^{-1} .$$
The \((r,s)\) element of \(\gamma^{-1} E_\theta^-(ZZ' - \sum -1)\) is then,

\[
(36) \quad E_\theta^-(Z^r Z^s - \sigma^{rs}) = \sum_{j=1}^{n} \psi_j \sum_{\ell=1}^{w} (\sigma^{\ell} s, \ell + j - 1 + \sigma^{r}, \ell + j - 1, s) \psi_j .
\]

Thus from (19), (21b), and (36) it follows that \(E_\theta^Y_k(Z) = \gamma \sum_{j=1}^{n} d_{kj} \psi_j\) and therefore

\[(37) \quad E_\theta^Y(Z) = \gamma D \psi .\]

Now \(\int z n(z) \sum -1 \varphi, \sum -1)dz = \sum -1 \varphi\) for any \(\varphi\). Thus from (7)

\[
E_\theta Z = \gamma \sum_{j=1}^{n-1} \left( \sum_{j=1}^{n-1} S_j \theta \right)
= \gamma \left( \sum_{-w}^{1} l_w \right) \psi_o .
\]

Thus \(E_\theta Z' l_w = \gamma \left( \sum_{-w}^{1} l_w \right) \psi_o .\)

Lemma 6. Let \(C\) be the covariance matrix of \((Y(Z)', Z' l_w)\). Then

\[
C = \begin{pmatrix}
D & \gamma O(\psi_o) \\
\gamma O(\psi_o) & l_w \sum -1 l_w
\end{pmatrix} + \gamma O(\|\theta\|^2)
\]

Proof. \(E_\theta [Y(Z) - \gamma D \psi][Y(Z)' - \gamma \psi' D]
= E_\theta Y(Z) Y(Z)' - \gamma^2 D \psi \psi' D .\)

From (20) and (21a), we have

\[
E_\theta^Y(Z) Y(Z)' = E_\theta Y(Z) Y(Z)' + \gamma O(\|\theta\|^2)
= D + \gamma O(\|\theta\|^2) .
\]

From (36) it is easy to show that
where $H^*$ is a $w \times n$ matrix with elements

$$h^*_{ij} = \sum_{s=1}^{w} \sum_{k=1}^{w} \left( \sigma_{i,k} s, i + j - 1 + \sigma_{i,k} s, s, s, \right).$$

We then have the indicated variance term for $Z' l_w$.

$$E_{\theta}(Y(Z) - \gamma D \psi)[Z' l_w - \gamma (l_w \sum l_w \psi_o)]$$

$$= E_{\theta} Y(Z) (Z' l_w) - \gamma^2 (l_w \sum l_w \psi o) D \psi_o$$

$$= \gamma [E_{\theta} (Z' l_w)^2 Y(Z) \psi_o + O(\|\theta\|^2)]$$

from (20). We thus have the desired result.

In the beginning of Section II we introduced the concept of asymptotic efficiency. However, in this section we will use a somewhat weaker concept of efficiency. A sequence of estimators $\{T_m\}$ will be called efficient near zero if for every vector $a$

$$\lim_{\gamma, \theta \to 0} a^T l_w (\varphi) a = 1.$$

Let $\overline{Y(Z)} = \frac{1}{m} \sum_{k=1}^{m} Y(Z^{(k)})$ where $Z^{(k)} = \sum l_w (k)$. Then we have

**Theorem 3.** If $w \geq n$, $\overline{\psi} = \gamma^{-1} D^{-1} \overline{Y(Z)}$ is an unbiased estimator of $\psi$ but it is not asymptotically efficient near zero.

**Proof.** From the central limit theorem and Lemmas 5 and 6

$$\sqrt{n} [\overline{Y(Z)} - \gamma D \psi] \to N(0, K_\psi)$$
where $K_y = D + \gamma O(\|\theta\|^2)$ is the covariance of $y$. Hence

$$
L[\sqrt{m} (\gamma - \psi)] \rightarrow \mathcal{N}(0, K_{\psi}) \text{ as } m \to \infty,
$$

where

$$
K_{\psi} = \gamma^{-2}D^{-1}K_yD^{-1}
$$

$$
= \gamma^{-2}[D^{-1} + \gamma O(\|\theta\|^2)]
$$
is the asymptotic covariance matrix of $\tilde{\psi}$. From Corollary 1.2, for any vector $a$

$$
a' \sum_{\psi} a = \gamma^{-2}[a'D^{-1}a - \frac{(a'D^{-1}n)^2}{n'D^{-1}n}] + o(1) + o(\|\theta\|^2).
$$

Thus

$$
\frac{a' \sum_{\psi} a}{a' K_{\psi} a} = 1 - \frac{(a'D^{-1}n)^2}{(n'D^{-1}n)(a'D^{-1}a)} + o(1) + o(\|\theta\|^2)
$$

and therefore,

$$
\lim_{\gamma, \theta \to 0} \frac{a' \sum_{\psi} a}{a' K_{\psi} a} < 1.
$$

This lack of efficiency is surprising since $\tilde{\psi}$ is a linear function of the sample autocorrelation $Y(Z)$ and one could expect that it would be efficient in estimating the waveform autocorrelation $\psi$. However, if $w < n$, we do have this "naive" estimator efficient in the estimation of the first $w$ partial correlations $\psi_1, \ldots, \psi_w$. We have

**Theorem 4.** If $w < n$, then $\tilde{\psi}(w) = \gamma^{-1}D^{-1}_w Y(Z \mid w)$ is an asymptotically efficient estimator near zero, of $\psi(w)' = (\psi_1, \ldots, \psi_w)$ where

$$
Y(Z \mid w)' = (Y_1(Z), \ldots, Y_w(Z)).
$$
Proof. By applying the central limit theorem in a manner similar to that used in the proof of Theorem 3, we have from (21a) and Lemmas 5 and 6,

\[ \mathcal{L}(\sqrt{m} [\hat{\psi}(w) - \psi(w)]) \to \mathcal{N}(0, K_{\psi(w)}) \text{ as } m \to \infty, \]

where

\[ K_{\psi(w)} = \gamma^{-2}[D^{-1}_w + \gamma O(\|\varepsilon\|^2)] \]

is the asymptotic covariance matrix of \( \hat{\psi}(w) \). Comparing \( K_{\psi(w)} \) with the lower bound for \( \sum \psi(w) \) as given in Corollary 2.2, we have the desired result.

Let us digress and derive an efficient estimator of \( \psi_o \). This estimator will be part of the efficient estimator of \( \psi \) when \( w \geq n \).

**Theorem 5.** \( \hat{\psi}_o = \gamma^{-1}(l_w^{-1} \sum l_w^{-1} \bar{z}^t \bar{l}_w \bar{l}_w) \) is an asymptotically efficient estimator of \( \psi_o \) near zero.

**Proof.** From Lemmas 5 and 6 and the central limit theorem,

\[ \mathcal{L}(\sqrt{m} (\hat{\psi}_o - \psi_o)) \to \mathcal{N}(0, k_o) \text{ as } m \to \infty, \]

where

\[ k_o = \gamma^{-2}[(l_w^{-1} \sum l_w^{-1})^{-1} + \gamma O(\|\varepsilon\|^2)] \]

is the asymptotic variance of \( \hat{\psi}_o \). Comparing \( k_o \) with \( \sigma^2_{\psi_o} \) as given by Corollaries 1.1 and 2.1, we have the desired result.

By combining \( \overline{Y(Z)} \) and \( \hat{\psi}_o \) we can obtain an efficient estimator of \( \psi \) when \( w \geq n \).
Theorem 6. If \( w > n \), then

\[
\hat{\psi} = \gamma^{-1}(D^{-1} - \frac{D^{-1} n' D^{-1} n}{n' D^{-1} D^{-1} n}) Y(Z) + \frac{D^{-1} n}{n' D^{-1} D^{-1} n} \frac{1}{2} (\psi_0)^2
\]

is an asymptotically efficient estimator of \( \psi \) near zero.

Proof. Let \( T_m' = (\overline{Y(Z)} , \overline{Z' l_w}) \). From Lemmas 5 and 6 and the central limit theorem,

\[
\sqrt{m} \left( T_m' - \gamma \left( \frac{D \psi}{n' \sum l_w' l_w \psi_0} \right) \right) \rightarrow \mathcal{N}(0, C) \text{ as } m \rightarrow \infty.
\]

Now let \( \varphi' = (\varphi_1, \ldots, \varphi_{n+1}) \) and define a function \( g \) such that

\[
g(\varphi) = H \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_n \\ \varphi_{n+1} \end{pmatrix} + h \varphi_{n+1}^2
\]

\[
H = \gamma^{-1} \left( D^{-1} - \frac{D^{-1} n' D^{-1} n}{n' D^{-1} D^{-1} n} \right),
\]

\[
h = \frac{D^{-1} n}{n' D^{-1} D^{-1} n} \frac{1}{2} \gamma^{-2} \left( l_w' \sum l_w' l_w \right)^{-2}.
\]

Thus

\[
g(T_m') = H \overline{Y(Z)} + h(\overline{Z' l_w})^2 = \hat{\psi}.
\]

We refer to the discussion of the delta method in Section II. We substitute
\[ g(\varphi^*) = \frac{1}{2} \gamma D \psi + h \gamma \frac{1}{\sum_{w} l_w} \psi_0^2 \]
\[ = \psi - \frac{D^{-1} n}{l_n} l'_n \psi + \frac{D^{-1} n}{l_n} \frac{1}{2} \psi_0^2 \]
\[ = \psi \]

since \( \frac{1}{2} \psi_0^2 = l'_n \psi \). Then

\[ L(\sqrt{m}(\hat{\psi} - \psi)) \to \mathcal{N}(0, K_{\psi}) \text{ as } m \to \infty \]

where the asymptotic covariance of \( \hat{\psi} \) is given by,

\[ K_{\psi} = \frac{\partial g}{\partial \psi} \bigg|_{0} \bigg[ \frac{\partial g}{\partial \varphi} \bigg] \]

\[ = \left[ H, 2h \gamma \left( l'_n \sum_{w} l_w \psi_0 \right) \right] C \left[ \begin{array}{c} H \\ 2h' \gamma \left( l'_n \sum_{w} l_w \psi_0 \right) \end{array} \right] \]

\[ = \gamma^{-2} \left[ D^{-1} - \frac{D^{-1} l_n l_n'}{l_n' D^{-1} l_n} + O(||\psi||^2) \right]. \]

Comparing \( K_{\psi} \) with \( \sum_\psi \) in Corollary 1.2, we see that \( \hat{\psi} \) is asymptotically efficient.

In the special case that \( w = n-1 \), using Corollary 2.3 we can prove as in the proof of Theorem 6,

**Theorem 7.** If \( w = n-1 \), then

\[ \hat{\psi}_n = \frac{1}{2} \left( \hat{\psi}_0^2 \right) - \gamma^{-1} D^{-1} \frac{1}{\gamma(Z|w)} \]

is an asymptotically efficient estimator of \( \psi_n \) near zero.
V. Examples.

In this section we shall give examples for some of the results in Sections III and IV for three cases: (A) white or uncorrelated noise, (B) white noise with \( n=w=2 \), and (C) Gaussian Markov noise with \( n=w \).

Example A. If the noise \( N(t) \) is white, the covariance matrix \( \Sigma \) is simply the \( w \times w \) identity matrix, \( I_w \), since we have made the normalization \( \sigma^2_{N(t)} = 1 \). Thus \( Z = X \) and

\[
Y(X)' = \left( \sum_{k=1}^{w} (X_k^2 - 1), \sum_{k=1}^{w-1} X_k X_{k+1}, \ldots, \sum_{k=1}^{w-n+1} X_k X_{k+n-1} \right).
\]

If \( w \geq n \),

\[
D = \mathbb{E}_\omega Y(X)Y(X)' = \begin{pmatrix}
2w & 0 \\
& \ddots \\
0 & & \ddots \\
& 0 & & \ddots \\
& & & \ddots \\
& & & & 0
\end{pmatrix}.
\]

If \( w < n \),

\[
D_w = \mathbb{E}_\omega Y(X|w)Y(X|w)' = \begin{pmatrix}
2w & 0 \\
& \ddots \\
0 & & \ddots \\
& 0 & & \ddots \\
& & & \ddots \\
& & & & 1
\end{pmatrix}.
\]

From Lemma 2 we have,

\[
I(\theta) = \gamma^2 J \left[ \frac{w}{\psi_0} \right] l_{1} l_{1}' + D + O(\gamma) + O(||\theta||^2) J',
\]

since \( l_{1}' \sum_{w} l_{1} l_{1}' = l_{1} l_{1}' = w \). Thus if we have \( w \geq n \), from Corollary 1.2,
\[ \sum_{\psi} = \gamma^{-2} \left[ \begin{pmatrix} \frac{1}{2^w} & 0 \\ \frac{1}{w-1} & \frac{1}{w-1} \\ \vdots & \vdots \\ 0 & \frac{1}{w-n+1} \end{pmatrix} \right] \]

\[ - \left[ \frac{1}{2^w} + \sum_{k=w-n+1}^{w-1} k^{-1} \right]^{-1} \left( \begin{pmatrix} \frac{1}{2^w} \\ \frac{1}{w-1} \\ \vdots \\ \frac{1}{w-n+1} \end{pmatrix} \right) \]

\[ \left( \frac{1}{2^w}, \frac{1}{w-1}, \ldots, \frac{1}{w-n+1} \right) \]

\[ + o(\gamma) + o(\|\theta\|^2) \]

Example 2. Let \( w=n=2 \)

\[ I(\theta) = \gamma^2 J_2 \left[ 2 \psi \begin{pmatrix} \frac{1}{2^w} \\ \frac{1}{w-1} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \right] + o(\gamma) + o(\|\theta\|^2) J_2^t \]

where

\[ J_2 = \begin{pmatrix} \theta_1 & \theta_2 \\ \theta_2 & \theta_1 \end{pmatrix} \]

Thus

\[ \sum_{\psi} = \gamma^{-2} \left[ \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & 1 \end{pmatrix} - \frac{4}{5} \begin{pmatrix} \frac{1}{16} & \frac{1}{4} \\ \frac{1}{4} & 1 \end{pmatrix} + o(\gamma) + o(\|\theta\|^2) \right] \]

\[ = \frac{1}{5\gamma^2} \left[ \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + o(\gamma) + o(\|\theta\|^2) \right] \]

\[ \hat{\psi}_o = \gamma^{-1} \left( \frac{1}{w} \sum_{-1}^{-1} (\bar{x}_1 \bar{x}_2) \right) = \frac{1}{2\gamma} (\bar{x}_1 + \bar{x}_2) \]

\[ \bar{x}_i = \frac{1}{m} \sum_{k=1}^{m} x_{i(k)}, \ i=1,2. \]

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\[
\hat{\psi} = \begin{pmatrix} \hat{\psi}_1 \\ \hat{\psi}_2 \end{pmatrix} \quad \text{where,}
\]

\[
\hat{\psi}_1 = \frac{1}{5\gamma} \left( \frac{x_1^2 - 1}{x_1^2} + \frac{x_2^2 - 1}{x_2^2} - x_1 x_2 \right) + \frac{1}{10} \hat{\psi}_0^2
\]

\[
\hat{\psi}_2 = \frac{1}{5\gamma} \left( x_1 x_2 - \frac{x_1^2 - 1}{x_1^2} - \frac{x_2^2 - 1}{x_2^2} \right) + \frac{2}{5} \hat{\psi}_0^2
\]

and

\[
\frac{x_i^2 - 1}{x_i^2} = \frac{1}{m} \sum_{k=1}^{m} (x_i^{(k)}^2 - 1) \quad \text{for } i=1,2
\]

\[
\frac{x_1 x_2}{x_1^2} = \frac{1}{m} \sum_{k=1}^{m} x_1^{(k)} x_2^{(k)}
\]

**Example C.** Let \( N(t) \) be Gaussian Markov noise with \( \mathbb{E} N(t+\tau)N(t) = \rho |\tau|, \quad 0 < \rho < 1. \)

Then

\[
\sum = \begin{pmatrix}
1 & \lambda & \lambda^2 & \lambda^3 & \ldots & \lambda^w \\
\lambda & 1 & \lambda & \lambda^2 & \ldots & \lambda^{w-1} \\
\lambda^2 & \lambda & 1 & \lambda & \ldots & \lambda^{w-2} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\lambda^w & \lambda^{w-1} & \lambda^{w-2} & \lambda^{w-3} & \ldots & 1
\end{pmatrix}
\]

where \( \lambda = \rho^T \) with \( T \) being the pulse width.

\[
(39) \quad \sum^{-1} = \frac{1}{1-\lambda^2} \begin{pmatrix}
1 & -\lambda \\
-\lambda & 1+\lambda^2 & -\lambda \\
-\lambda & 1+\lambda^2 & -\lambda & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots \\
0 & \ldots & \ldots & 1+\lambda^2 & -\lambda \\
-\lambda & \ldots & \ldots & \ldots & 1
\end{pmatrix}
\]
Let \( w = n > 3 \).

We have for \( D = (d_{ij}) \), from (21b)

\[
\begin{align*}
d_{i, i+3} &= 0 \quad \text{for } i = 1, \ldots, n-3 \\
d_{i, i+2} &= \begin{cases} 
(1-\lambda^2)^{-2}(n-i-1)\lambda^2 & \text{for } i = 2, \ldots, n-2 \\
(1-\lambda^2)^{-2}2(n-2)\lambda^2 & \text{for } i = 1 
\end{cases} \\
d_{i, i+1} &= \begin{cases} 
-(1-\lambda^2)^{-2}2\lambda[(n-1) + (n-1)\lambda^2] & \text{for } i = 2, \ldots, n-2, n-1 \\
-(1-\lambda^2)^{-2}4\lambda[(n-1) + (n-2)\lambda^2] & \text{for } i = 1 
\end{cases} \\
d_{ii} &= \begin{cases} 
(1-\lambda^2)^{-2}2[n+2(2n-3)\lambda^2 + (n-2)\lambda^4] & \text{for } i = 1 \\
(1-\lambda^2)^{-2}[(n-1) + (5n-9)\lambda^2 + (n-3)\lambda^4] & \text{for } i = 2 \\
(1-\lambda^2)^{-2}[(n-i+1) + 4(n-i)\lambda^2 + (n-i-1)\lambda^4] & \text{for } i = 3, \ldots, n-1 \\
(1-\lambda^2)^{-2} & \text{for } i = n 
\end{cases}
\end{align*}
\]
VI Estimation of $\theta$.

We have shown that $\hat{\psi}$ and $\hat{\psi}_o$ are asymptotically efficient estimators of $\psi' = (\psi_1, \ldots, \psi_n)$ and $\psi_o$ respectively. Unfortunately the problem of estimating $\theta' = (\theta_1, \ldots, \theta_n)$ is much more difficult. In this section we will discuss informally a method for estimating $\theta$, based upon the estimates $\hat{\psi}$ and $\hat{\psi}_o$. However, this method seems not very practical to implement.

Recall that $\psi$ is defined by the system of second order polynomials in the $\theta_i$'s as follows:

\begin{equation}
(1) \quad \psi_1 = \frac{1}{2} \sum_{i=1}^{n} \theta_1^2, \quad \psi_2 = \sum_{i=1}^{n-1} \theta_1 \theta_{i+1}, \ldots, \psi_n = \theta_1 \theta_n
\end{equation}

with $\psi_o = \sum_{i=1}^{n} \theta_i$. Remember that $\theta$ is in a region such that $J$, the Jacobian matrix of the transformation, does not vanish and thus $J^{-1}$ exists. However, given $\psi_o, \psi_1, \ldots, \psi_n$ there is not a unique solution to (1).

For simplicity suppose that $\psi_o$ and $\psi$ are known and $\theta^{(1)}, \theta^{(2)}, \ldots, \theta^{(r)}$ are the corresponding real solutions. Then, let

$$\beta^{(k)} = X_i \theta^{(k)}.$$

With each observation, depending on the position of the vector $\theta$ in the "window" $X$, each of the $r$ random variables $\beta^{(k)}$ are normally distributed. In fact $\beta^{(k)}$ has mean 0 with probability $1-(n+w-l)\gamma$, or $\mu_{kj} = (X_j \theta)' \theta^{(k)}$ with probability $\gamma$ for each $j=w+1, \ldots, n-1$. Moreover given any position, $\beta^{(k)}$ has variance $\theta^{(k)}' \sum \theta^{(k)}$ which
is a constant
\[ \sigma^2 = 2[\psi_1 \sigma_{11} + \psi_2 \sigma_{12} + \cdots + \psi_n \sigma_{1n}] \]
if \( \sum \) corresponds to stationary noise, i.e., if \( \sigma_{ij} = \sigma_{i+1,j+1} \).

The Cauchy-Schwarz Inequality informs us that \( \mu_{k,j} \) achieves a unique maximum \( 2\psi_1 \) when \( j = 0 \) and \( \theta^{(k)} = \theta \). Thus if we set a limit
\[ c = 2\psi_1 + \lambda \sigma, \]
the proportion of observations \( \beta^{(k)} \) exceeding \( c \) is
\[ p_k(\lambda) = \sum_j \left[ 1 - \Phi[\lambda + 2(\psi_1 + \mu_{k,j})] \right] \]
\[ + \left[ 1 - (n+1)\gamma \right] \Phi[\lambda + 2\psi_1] \]
where \( \Phi \) is the c.d.f. of a standard normal variable.

By making \( \lambda \) large enough, the largest \( p_k(\lambda) \) corresponds to the \( k \) for which \( \theta^{(k)} = \theta \). Let the selection procedure be to select that \( \theta^{(k)} \) for which the corresponding \( \beta^{(k)} \) exceeds \( c \) the most times in the \( m \) observations. This procedure produces \( \theta \) with probability approaching one providing (i) \( \lambda \) is sufficiently large depending on \( m, \gamma \), and \( \mu_{k,j} \); (ii) \( m \to \infty \) and is sufficiently large depending on \( \gamma \) and the \( \mu_{k,j} \).

However, we do not know \( \psi \) and \( \psi_o \), but we have the estimates \( \hat{\psi} \) and \( \hat{\psi}_o \). From Theorems 4 and 5, \( \hat{\psi} \to \psi \) and \( \hat{\psi}_o \to \psi_o \) in probability as \( m \to \infty \). Then in (1), for large \( m \), replace \( \psi \) by \( \hat{\psi} \) and also \( \psi_o \) by \( \hat{\psi}_o \), and obtain the real solutions \( \hat{\theta}^{(1)}, \ldots, \hat{\theta}^{(r)} \). As \( m \to \infty \) each \( \hat{\theta}^{(k)} \to \theta^{(k)} \) in probability. The above selection procedure will then select that \( \hat{\theta}^{(k)} \) such that \( \hat{\theta}^{(k)} \to \theta \) in probability as \( m \to \infty \).

As yet we have not given a method for determining the real solutions of (1). As a possible contribution toward finding and counting the
\( \theta^{(k)} \), we now present a lemma which describes the real \( \theta^* \) corresponding to \( (\Psi, \Psi) \).

**Lemma 7.** Given \( (\Psi, \Psi) \), a real solution \( \theta^* \) of the system of equations given by (1) with \( \Psi = \sum_{i=1}^{n} \theta^* \), has the form

\[
\theta^* = \frac{1}{n} A \varphi
\]

where \( \varphi \) is a (complex) vector whose first coordinate is \( \Psi \), and such that

\[
|\varphi_j| = \left| 2 \sum_{i=1}^{n} \psi_i \cos 2\pi \frac{(i-1)(j-1)}{n} \right|^{1/2}
\]

\[
\bar{\varphi}_{n-j+2} = \varphi_j \quad j=2,3,\ldots, n
\]

and with \( \rho = e^{2\pi i/n} \)

\[
A = \begin{pmatrix}
1 & 1 & 1 & \ldots & 1 \\
1 & \rho & \rho^2 & \ldots & \rho^{n-1} \\
1 & \rho^2 & \rho^4 & \ldots & \rho^{2(n-1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \rho^{n-1} & \rho^{2(n-1)} & \ldots & \rho^{(n-1)(n-1)}
\end{pmatrix}
\]

**Proof.** Applying (1) we have

\[
\left| \sum_{k=1}^{n} \theta_k e^{2\pi i (k-1)\omega} \right|^2 = 2 \sum_{k=1}^{n} \psi_k \cos 2\pi (k-1)\omega.
\]

Setting \( \omega = \frac{1-1}{n} \) in (41), with \( j=1,\ldots,n \) it follows that the absolute value of the \( j^{th} \) component of \( \varphi = A\theta^* \) is

\[
2 \left| \sum_{i=1}^{n} \psi_i \cos 2\pi \frac{(i-1)(j-1)}{n} \right|^{1/2}.
\]
Observe that \( \varphi_1 = \psi_0 \). Since the \( n \) th roots of unity add to zero, we have

\[
A^{-1} = \frac{1}{n} \overline{A}.
\]

Thus we have (40). Moreover since \( \theta^* \) must be real,

\[
\theta^* = \frac{1}{n} \overline{A} \varphi = \frac{1}{n} A \overline{\varphi}.
\]

This implies \( \varphi = \frac{1}{n} A^2 \overline{\varphi} \). We can check that the \( j \) th row of \( A^2 \) has zeros in all columns except for a one in the \((n-j+2)\)th column. Thus we have that \( \varphi_j \overline{\varphi} = \varphi_{n-j+2} \), \( j \geq 2 \).

For example if \( n=3 \),

\[
\begin{align*}
3\theta^*_1 &= \psi_0 + 2(2\psi_1 - \psi_2 - \psi_3)^{1/2} \cos 2\pi \omega \\
3\theta^*_2 &= \psi_0 + 2(2\psi_1 - \psi_2 - \psi_3)^{1/2} \cos 2\pi (\omega + \frac{1}{3}) \\
3\theta^*_3 &= \psi_0 + 2(2\psi_1 - \psi_2 - \psi_3)^{1/2} \cos 2\pi (\omega - \frac{1}{3})
\end{align*}
\]

where \( 0 \leq \omega < 1 \). Observe that \( 2\psi_1 - \psi_2 - \psi_3 \geq 0 \). However, we do not obtain a solution of (1) for all \( \omega \), only for a finite number of \( \omega \). By solving (1) for \( n=3 \) in a straight-forward manner, it turns out that there are four possible real solutions and thus there are four \( \omega \) (functions of \( \psi \)) which give \( \theta^*_1, \theta^*_2, \) and \( \theta^*_3 \) as real solutions.

We conjecture that there are \( 2^{n-1} \) possible real solutions to (1) such that \( \psi_0 = \sum_{k=1}^{n} \theta^*_k \).

Finally, we must admit that in the formulation of the statistical problem we have left out certain important aspects of the PPM communication system. The informed receiver \( B \) in the system has certain
a-priori information, in addition to $\theta$ and $\gamma$, at his disposal in order to detect the recurrences of the waveform, measure the intervals between recurrences, and decode the message which $A$ has sent out.

However, we feel that there are three basic ways in which this work is a contribution to the PFM detection problem. First, it gives a statistical formulation in terms of sampling, estimation, and detection of unknown parameters. Secondly, it shows that given the formulation, the basic parameter is the autocorrelation vector of the waveform vector and the basic statistic is the sample autocorrelation. Thirdly, using the tools of large-sample theory, it develops an asymptotically efficient estimator of the discrete autocorrelation. The estimator is fairly simple and it is hoped that it has good properties even if $\gamma$ is near one and the signal-to-noise ratio is not small.
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