ON SEQUENTIAL DESIGNS FOR
MAXIMIZING THE SUM OF \( n \) OBSERVATIONS

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1. Introduction.

Certain aspects of the type of problem known as the 'Two-armed Bandit' are considered. In its 'classical' formulation, whence the name, we have a slot machine with two arms, an X arm and a Y arm. Upon pulling either arm the machine pays off either one unit or nothing and the probability of winning with one arm is p and for the other, q. A priori it is unknown which is which but the probability $\xi$ that it is the X arm which has probability p of success is assumed known. One is allowed n plays and a sequential design, or strategy, is desired which will maximize the expected winnings.

Leaving the machine terminology, we have two binomial random variables, X and Y, having parameters under the two hypotheses, $H_1$ and $H_2$, given by

\[
\begin{pmatrix}
\xi \\
1-\xi
\end{pmatrix}
\begin{pmatrix}
H_1 \\
H_2
\end{pmatrix}
\begin{pmatrix}
p \\
q
\end{pmatrix}
\]

where $\xi$ is the a priori probability that $H_1$ is true. We wish to maximize the sum of n observations.

Rather more realistic for many problems is to leave the dichotomous
situation and to suppose that $X$ and $Y$ have parameters $p$ and $q$, respectively, and that an a priori distribution $F(p,q)$ is known. The problem holds interest for several reasons. It has been proposed and not yet answered. It would appear to be one of the simplest problems in the sequential design of an experiment that can be put; hence its analysis is a step towards obtaining a body of information relative to specific sequential design problems. It has not only this general interest but, as it stands, has applications in particular problems such as learning theory, biology, and medicine; see [1], for instance, in which applications in the latter two fields may be found. A discussion of problems of this general variety and of certain strategies has been published by Robbins [1]. More immediately, in the final section of this paper it is shown that the solution to the problem in which $p$ has a priori distribution $F$ and $q$ is assumed known, explicitly obtained in Section 4, yields directly the solution of a problem in industrial inspection.

We shall use here for intuitive concreteness the gambling interpretation and terminology.

It has been conjectured that for the problem as first put, the optimal strategy is $S_1$: on each play choose the arm having, at that time, the maximum expected probability of paying off, i.e., play each time as though there were but one play remaining. This conjecture has been verified to hold for $n \leq 6$.

The original problem can be generalized in two directions. The random variables may have distributions other than binomial. Sufficient conditions that $S_1$ be optimal are given and it is shown that for the binomial case following $S_1$ the expected winnings per play tends to max $[p,q]$ as $n \rightarrow \infty$. 
The other direction of generalization was mentioned above, $X$ and $Y$ are binomial with parameters $p$ and $q$ having a priori distribution $F(p, q)$. In Section 3 it is shown that several properties which one intuitively expects the optimum strategy to possess are not, in general, characteristics of the optimal strategy; e.g., for $p$ and $q$ independently distributed, if only $n$ is sufficiently large $S_1$ is not optimal; the optimal strategy may not stay on a winner; and the expected winnings on the $r^{th}$ play is not necessarily a non-decreasing function of $r$. Also, $S_k$, the strategy which maximizes the expected winnings over the next $k$ plays, is not always an improvement over $S_{k-1}$.

In Section 4 the parameter $q$ and the a priori distribution $F(p)$ are assumed known. In this case the optimal strategy is determined explicitly and is shown to have those intuitive properties which have previously been noted not to be general characteristics of optimal strategies. These results are applied, in the final section, to obtain the optimal procedure in a certain industrial inspection problem.
2. The 'Two-armed Bandit'.

2.1. The statistical problem which goes under this general title is that of finding a design which will maximize the sum of $n$ independent observations in the following situation: let $X$ and $Y$ be real valued random variables having c.d.f.'s $F_1$ and $G_1$, respectively, under hypothesis $H_i$ ($i = 1, 2$) and $\xi$ be the a priori probability that $H_1$ is the true hypothesis. The problem is to devise a sequential design which will maximize the expected value of the sum of $n$ observations, each of which is to be an observation either of $X$ or $Y$.

Let $f_1$ and $g_1$ be the densities corresponding to $F_1$ and $G_1$ with respect to the measure $\Psi$. Let $W_n(\xi, S^*)$ denote the expected value of the sum of the $n$ observations if $\xi$ is the a priori probability for $H_1$ and the design, $S^*$, is used. If one observed $X$ first and then continued for $n-1$ steps following the optimal rule $S^*$, then the expected sum would be

$$2.1.1 \quad A_n = \xi \int_{-\infty}^{\infty} tf_1(t) d\Psi + (1 - \xi) \int_{-\infty}^{\infty} tf_2(t) d\Psi$$

$$+ \int_{-\infty}^{\infty} \frac{sf_1(t)}{\xi f_1(t) + (1 - \xi)f_2(t)} (s^* (1 - \xi) f_2(t)) d\Psi.$$

Similarly, if $Y$ were observed first and the optimal rule followed for the remaining $n-1$ steps, the expected sum would be
(2.1.2) \[ B_n = \int_{-\infty}^{\infty} t g_1(t) d \psi + (1-\gamma) \int_{-\infty}^{\infty} t g_2(t) d \psi \]

\[ + \int_{-\infty}^{\infty} \frac{S_2(t)}{S_1(t) + (1-\gamma) g_2(t)} S_n(t) (S_2(t) + (1-\gamma) g_2(t)) d \psi. \]

Hence, \( W_n(S, S_n) = \max (A_n, B_n) \).

A natural design to be considered is that which requires that one maximize step by step, i.e., after the \( j \)th observation the \( \text{a posteriori} \) probability, \( S_j \), is computed and at the next step observe the random variable corresponding to the maximum of \( \int_{-\infty}^{\infty} t(S_j f_1(t) + (1-\gamma) f_2(t)) d \psi \)

and \( \int_{-\infty}^{\infty} t(S_j g_1(t) + (1-\gamma) g_2(t)) d \psi \). Denote this stepwise maximization design by \( S_1 \).

**Theorem 2.1.** If the likelihood ratios \( f_2/f_1 \) and \( g_2/g_1 \) have the same distributions under \( H_1 \) and also under \( H_2 \), then \( S_1 \) is the optimal design.

**Proof.** Since

\[
S_1 = \begin{cases} 
\frac{1}{1 + \frac{1-\gamma}{\gamma} \frac{f_2(x)}{f_1(x)}} & \text{if } X \text{ is observed first,} \\
\frac{1}{1 + \frac{1-\gamma}{\gamma} \frac{g_2(x)}{g_1(x)}} & \text{if } Y \text{ is observed first,}
\end{cases}
\]
and the likelihood ratios have the same distributions, the distribution of $\zeta_1$ is independent of which random variable is observed first. Hence, the expected value of the optimal yield from the last $n-1$ steps is independent of the choice for the first step. One can, therefore, maximize the expected sum of $n$ observations by choosing at the first step the random variable having the larger expected value and continuing with the optimal design for the remaining steps.

Since all the random variables are assumed to be independent, the same argument shows that, given $\zeta_j$, it is optimal to follow $S_{1j}$ for the $j+1$st step.

An example in which the likelihood ratios are distributed alike is:

$$
\begin{array}{ccc}
  X & Y \\
  H_1 & N(\mu_0,1) & N(\mu,1) \\
  H_2 & N(\mu,1) & N(\mu_0,1) & (\mu > \mu_0, \mu > 0),
\end{array}
$$

with $\mu = \mu_0$. However, it can be shown that for $n = 2$ and $(1-\delta)\mu = \delta \mu_0$, $S_1$ is optimal only if $\mu = \mu_0$.

2.2 A special case of the Two-armed-Bandit of widespread interest, the "classical" case, is that in which the random variables have binomial distributions with parameters given by:

$$
\begin{array}{ccc}
  X & Y \\
  H_1 & p & q \\
  H_2 & q & p
\end{array}
$$
A second example in which the likelihood ratios are distributed alike is furnished here if \( p + q = 1 \). Hence, for that case, \( S_1 \) is the optimal design. Indeed, it is a conjecture that for any choice of \( p \) and \( q \), \( S_1 \) is optimal; it has been verified to be for \( n \leq 3 \). Optimal or not, \( S_1 \) has the desirable property of being consistent, i.e.,

**Theorem 2.2.** Following the design \( S_1 \), the expected value of the average of the first \( n \) observations converges to \( \max \{ p, q \} \) as \( n \to \infty \).

**Proof.** Assume \( p > q \). Then

\[
W_1(\xi, S_1) = \begin{cases} 
q + (p-q)\xi & \text{for } \xi \geq 1/2 \\
p - (p-q)\xi & \text{for } \xi \leq 1/2 
\end{cases}
\]

and if \( \xi \geq 1/2 \),

\[
W_n(\xi, S_1) = W_1(\xi, S_1) \cdot W_{n-1}\left(\frac{p\xi}{\xi + 1}\right)^P \left(\frac{1-p}{\xi + 1}\right)^{1-P} \left(\frac{(1-q)^2\xi}{\xi^2}\right)^P \left(\frac{(q^2)^2\xi}{\xi^2}\right)^{1-P} \left(\frac{(1-q)^2\xi}{\xi^2}\right)^P \left(\frac{(q^2)^2\xi}{\xi^2}\right)^{1-P},
\]

while if \( \xi \leq 1/2 \),

\[
W_n(\xi, S_1) = W_1(\xi, S_1) \cdot W_{n-1}\left(\frac{p\xi}{\xi + 1}\right)^P \left(\frac{1-p}{\xi + 1}\right)^{1-P} \left(\frac{(1-q)^2\xi}{\xi^2}\right)^P \left(\frac{(q^2)^2\xi}{\xi^2}\right)^{1-P} \left(\frac{(1-q)^2\xi}{\xi^2}\right)^P \left(\frac{(q^2)^2\xi}{\xi^2}\right)^{1-P},
\]

where \( P_\xi(\xi = 0) = P_\xi(\xi = 1) \cdot (1-\xi)P_\xi(\xi = 2) \cdot (1-\xi)P_\xi(\xi = 3) \).

\( W_1 \) is clearly convex, symmetric about \( \xi = 1/2 \), and continuous. So is \( W_n \), since by an inductive argument \( W_n \) is symmetric about \( 1/2, \) (2) and (3) are continuous and each (by formal differentiation twice) is convex.

Also it is easily seen that \( W_n(\xi, S_1) = n \left[ (p-q)\xi + q \right] \) for \( \xi \) near 1.

Let \( a_n(\xi, S_1) = \frac{1}{n} W_n(\xi, S_1). \) Then \( a_n \) is convex, continuous, and bounded above by \( p \) on \([0,1]\). Furthermore, \( |a_n(\xi, S_1)| \leq p-q. \) As a
consequence of a more general result below. Lemma 3.3, \( a_n \) is non-decreasing in \( n \). Hence, \( a(\xi, S_1) = \lim_{n \to \infty} a_n(\xi, S_1) \) exists and is convex and continuous on \([0, 1]\). Moreover, since \( a_n(\xi, S_1) \) satisfies (1) and (2)

\[
\begin{align*}
(4) \quad a(\xi, S_1) &= \\
&= \begin{cases} \\
\frac{p \xi}{\xi (X=1)} P_{\xi} (X=1) + a \left( \frac{(1-p)\xi}{P_{\xi} (X=0)} \right) P_{\xi} (X=0), & \xi \geq 1/2 \\
\frac{q \xi}{\xi (Y=1)} P_{\xi} (Y=1) + a \left( \frac{(1-q)\xi}{P_{\xi} (Y=0)} \right) P_{\xi} (Y=0), & \xi \leq 1/2.
\end{cases}
\end{align*}
\]

Suppose that the minimum of \( a(\xi, S_1) \) is assumed at \( \xi_0 = 1/2 \). Then it also assumes its minimum at \( p \xi_0 / P_{\xi_0} (X=1) > \xi_0 \). By iteration, it assumes

\[
p^n \xi_0
\]

its minimum at

\[
p^n \xi_0 = q^n (1 - \xi_0)
\]

which tends to 1 as \( n \to \infty \). Hence, \( \xi_0 \) could be taken to be 1. If, on the other hand, \( \xi_0 < 1/2 \), the analogous procedure shows that \( \xi_0 \) could be taken to be 0. Thus, the minimum of \( a(\xi, S_1) \) is assumed either at 0 or 1. But \( a(0, S_1) = a(1, S_1) = p \), which establishes the theorem.

3. A Generalized 'Two-armed Bandit'. 1/

This section is concerned with the Bayes problem of maximizing the expected number of successes in \( n \) trials when at each trial we are free to choose between two binomial random variables, \( X \) and \( Y \), whose probabilities of success, \( p \) and \( q \), respectively, are unknown but a known

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1/ This section represents an extension of some preliminary work by S. Johnson and S. Karlin at RAND Corporation.
a priori distribution, $F(p, q)$, is specified.

The special case where $F(p, q)$ concentrates at the two fixed points $(p, q)$ and $(q, p)$ with probabilities $\xi$ and $1-\xi$, respectively, leads to the 'classical' problem considered in Section 2.2.

Let $S$ denote a strategy for choosing between $X$ and $Y$ and $W_n((p, q), S)$ denote the expected number of successes in following $S$ for $n$ plays for given $p$ and $q$. Then the expected number of successes is

$$W_n(F, S) = \frac{1}{2} \int_0^1 \int_0^1 W_n((p, q), S) \, dF(p, q).$$

We will find it convenient sometimes to express this as $W_n(dF, S)$. The best strategy is the one maximizing $W_n(F, S)$. Since $n$ is finite, the maximum exists.

Note that if $S$ is followed for $k-1$ plays and there have occurred $r_1$ successes and $s_1$ failures with $X$ and $r_2$ successes and $s_2$ failures with $Y$, then the conditional contribution on the $k$th step is

$$\frac{1}{2} p F(p, q \mid r_1, s_1, r_2, s_2) \text{ or } \frac{1}{2} q F(p, q \mid r_1, s_1, r_2, s_2)$$

according as $S$ calls for $X$ or $Y$ where

$$F(p, q \mid r_1, s_1, r_2, s_2) = \frac{\frac{r_1}{1-p} \frac{s_1}{q} \frac{r_2}{1-q} \frac{s_2}{q} \, dF(p, q)}{\int \int_0^1 \int_0^1 \frac{r_1}{1-p} \frac{s_1}{q} \frac{r_2}{1-q} \frac{s_2}{q} \, dF(p, q)}.$$

Example 1. Suppose $p + q = 1$ with probability $1$, i.e., $F(p, q)$ is of the form $F(p, 1-p)$. In this case a success or failure of $X$ is equivalent to (gives the same information as) a failure or success with $Y$, respectively.

$S_1$ is optimal in this case. For $1st$ $F_{k-1}(p, q)$ denote the a posteriori probability after $k-1$ plays and $S_{n-k}$ the optimal strategy for $n-k$ plays.
Then using X followed by \( S_{n-k} \) yields

\[
\int p^{F}_{k-1} \ast (W_{n-k}(p^{F}_{k-1}S_{n-k})) \int p^{F}_{k-1} \ast W_{n-k}((l-p)d^{F}_{k-1}S_{n-k}) \int (1-l)d^{F}_{k-1}
\]

and Y followed by \( S_{n-k} \) yields

\[
\int q^{F}_{k-1} \ast W_{n-k}(q^{F}_{k-1}S_{n-k}) \int q^{F}_{k-1} \ast W_{n-k}((l=q)d^{F}_{k-1}S_{n-k}) \int (1-q)d^{F}_{k-1}
\]

Since \( q = 1-p \), X is the optimal play if and only if

\[
\int p^{F}_{k-1} \geq \int q^{F}_{k-1}, \text{ i.e., } S_{1} \text{ is optimal. This example is related to the result of Theorem 2.1, which embraces a special case of } F(p,q) = F(p,1-p).
\]

\[2.1.\] Our first task is to obtain the complete strategy for \( n=2 \) when the a priori distribution is \( F(p)G(q) \). It is important to notice that if the number of trials is \( n \), then only designs which are functions of the first \( n \) moments, \( \mu_{1}, \ldots, \mu_{n} \) of \( F \) and \( \mu'_{1}, \ldots, \mu'_{n} \) of \( O \), need be considered. This is a consequence of the fact that the expected yield for any strategy is an expression involving at most these moments. Thus all strategies describing a first move can be expressed in terms of functions

\[
T_{j}(\mu_{1}, \ldots, \mu_{n}, \mu'_{1}, \ldots, \mu'_{n}) \text{ such that } T_{j}(\mu_{1}, \ldots, \mu_{n}, \mu'_{1}, \ldots, \mu'_{n}) \geq 0
\]

then X is chosen at the first trial, otherwise Y is used first.

Suppose for definiteness that \( \mu_{1} \geq \mu'_{1} \), we determine necessary and sufficient conditions that X be used first when \( n=2 \). Using the fact that on the last trial one chooses the random variable having greatest expected value, if X is used first the expected yield is

\[(1) \\mu_{1} + \mu'_{1} \left( \frac{\mu_{2}}{\mu_{1}} \right) = (1-\mu'_{1}) \max \left\{ \mu'_{1}, \frac{\mu_{1}}{1-\mu'_{1}} \right\} \].
Since \( (1) \geq 2 \mu_1 \geq \mu_1^* + \mu_1^* \geq 2 \mu_1^* \mu_1 \) X followed by optimal is better than Y followed always by X which is better than Y followed always by Y. Of the other two strategies starting with Y, the one requiring X if \( Y=1 \) has expected yield \( 2 \mu_1^* + \mu_1^* \mu_1^* = \mu_2^* \) which can be shown to be less or equal that for the strategy requiring Y if \( Y=1 \), namely

\[
(2) \quad \mu_1^* + \mu_2^* + (1-\mu_1^*) \mu_1^* .
\]

Upon comparison, \( (2) \leq (1) \) if and only if

\[
(3.1.1) \quad \text{either } \mu_2^* \geq \mu_2^* \text{ or } \mu_1^* + \mu_1^* \mu_1^* \geq \mu_1^* + \mu_2^* .
\]

Combining and rewriting in a symmetric form, we have

**Lemma 3.1.** If \( n=2 \) and \( p \) and \( q \) are independent with moments \( \mu_1 \) and \( \mu_1^* \), then X is used on the first trial if and only if

\[
\max \left\{ \mu_2^* - \mu_1^* \mu_1^* - \mu_1^* \right\} \geq \max \left\{ \mu_2^* - \mu_1^* \mu_1^* - \mu_1^* \right\} .
\]

Our next theorem shows that in almost all circumstances of independent \textit{a priori} distributions for \( p \) and \( q \), the optimal design and \( S_1 \) cannot agree.

**Theorem 3.2.** If \( p \) and \( q \) are independent with \textit{a priori} distributions \( F(p) = \int_p^0 \beta(t)dt \) and \( G(q) = \int_q^0 \psi(t)dt \) where \( \beta \) and \( \psi \) are continuous and positive for \( 0 < t < 1 \), then there exists an \( n \) such that for \( n \) trials the optimal design does not agree with \( S_1 \).

**Proof.** (By contradiction). Suppose for definiteness that \( 1 > \int_0^1 t \beta(t)dt = b > a = \int_0^1 t \psi(t)dt \). According to \( S_1 \) it is clear that X is used first.
and, by the Schwartz inequality, that we do not change random variables if a success occurs.

It is easily shown in view of the hypothesis on \( \phi \) that if \( r \) and \( s \) tend to infinity so that \( \frac{r}{r+s} \to t_0, \) then

\[
\begin{align*}
\frac{1}{r+s} t^{r+1} (1-t)^s \phi(t) dt & \to t_0, \\
\int_0^1 t^r (1-t)^s \phi(t) dt
\end{align*}
\]

(1)

(This can also be obtained as a consequence of the law of large numbers where the relative frequency of success tends to \( t_0 \)). Hence, taking \( t_0 = a + \epsilon, \) we can choose \( r \) and \( s \) so that

\[
\left| \frac{r}{r+s} = a \right| < \epsilon,
\]

(2)

\[
\begin{align*}
a + \epsilon > \frac{1}{r+s} t^{r+1} (1-t)^s \phi(t) dt & \to a, \\
\int_0^1 t^r (1-t)^s \phi(t) dt
\end{align*}
\]

\[
\begin{align*}
a + \epsilon > \frac{1}{r+s} t^{r+2} (1-t)^s \phi(t) dt & \to a, \\
\int_0^1 t^r (1-t)^s \phi(t) dt
\end{align*}
\]

Furthermore, \( \epsilon \) may be chosen sufficiently small that also

\[
\int t^2 \phi(t) dt > \left[ \int t \phi(t) dt \right]^2 + 3 \epsilon.
\]

Now let \( n = r + s + 2 \) and suppose that the first \( r \) plays resulted in successes with \( X_1 \) and the next \( s \) were failures with \( X_2. \) This agrees with the procedure prescribed by \( S_1 \) and has positive probability of occurrence.
There are now two plays left and, from (2), $S_1$ requires $X$ on the next step. However, (6.2.1) gives necessary and sufficient conditions that use of $X$ is optimal and we show that these are violated.

There are two steps left and the a posteriori probability distribution is $F(x)G(q)$ where

$$F(x) = \frac{p^x}{\int_0^x (1-t)^s \phi(t) dt}$$

and $G(q) = \int_0^q \psi(t) dt$.

On account of (2)

$$\int_0^1 p \, dF^1(p) > \int_0^1 q \, \psi(q) dq.$$

But

$$\int_0^1 p^2 \, dF^2(p) = \int_0^1 \frac{p^{x+2}}{p^x (1-p)^s \phi(p) dp} = \int_0^1 \frac{p^{x+2} (1-p)^s \phi(p) dp}{p^x (1-p)^s \phi(p) dp}$$

$$\leq (a+\varepsilon)^2 0 < a < 3 \leq \int_0^1 t \, \psi(t) dt.$$

Also,

$$\int_0^1 q \, \psi(q) dq < a + \varepsilon + a^2 + \varepsilon$$

$$< \int_0^1 t \, \psi(t) dt + \int_0^1 t^2 \, \psi(t) dt.$$

Hence, following $S_1$ we arrive at a non-optimal yield and the theorem is established.

3.2 $S_j$ may be described as that procedure requiring at each step the random variable which would be optimal were there but one trial remaining.

In a similar spirit let $S_j$ be the strategy which requires at each trial the
random variable which would be optimal were there j trials remaining with
the understanding that if fewer than j trials remain, then the optimal
procedure is followed. (The strategy S∗ for p and q independent is
determined by the relations given in Lemma 3.1).

We have, thus, a sequence of strategies S∗, S∗, S∗, . . . , S∗. For a series
of n trials, S∗ is the optimal strategy and hence Wn(F, Sn) > Wn(F, Sj)
for all j < n. Intuitively one might expect that the Wn(F, Sj) are
non-decreasing in j, i.e., the more steps ahead we take into account the
better the strategy. However, we show that there exist a priori distributions
such that for n = 3, W3(F, S1) > W3(F, S2).

Suppose p and q independent with distributions F and G and
moments µ1 and µ∗, respectively, and that

\[ \mu_1 > \mu_1^* \quad \mu_2 < \mu_2^* \quad \text{and} \quad \mu_3 > \mu_3^* \]

(3.2.1)

\[ \mu_1^* = \mu_1^*, \mu_1^* < \mu_1^* \}

\[ \mu_2^* = \mu_2^*, \mu_3^* \]

Upon determining W3(FG, S∗) and W3(FG, S∗) subject to these inequalities,
it is seen that W3(FG, S1) = W3(FG, S2) = \mu_3 - \mu_3^* > 0. Hence, S2 is
not necessarily an improvement on S1, provided there exist distributions
satisfying (3.2.1).

Let \( a_1, a_2, a_3 \) and \( b_1, b_2, b_3 \) denote the first three moments, respectively,
of any two distributions H and K on [0,1] such that \( a_1 > b_1, a_2 < b_2 \) \( \text{and} \)
\[ a_3 > b_3; \ e.g., a_1 = \frac{1}{2}, a_2 = \frac{1}{3}, a_3 = \frac{1}{4}, b_1 = \frac{1}{2} = \gamma, b_2 = \frac{1}{3} + \gamma, b_3 = \frac{1}{4} = \gamma. \]
which are, for sufficiently small $\gamma > 0$, moments of distributions on $[0,1]$. Let $L$ be any distribution on $[0,1]$ whose moments,

$$c_1, c_2, c_3, \text{ satisfy } c_1 > \frac{c_2 - c_3}{c_1 - c_2} ; \text{ so } L \text{ concentrates on } .9 \text{ and } .8.$$

The Schwartz inequality implies that $c_1 < \frac{c_1 + c_2}{1 + c_1}$. For $0 < \varepsilon < 1$, let

$$F(p) = \varepsilon F(p) + (1-\varepsilon) L(p) \text{ and } G(q) = \varepsilon G(q) + (1-\varepsilon) L(q). \text{ Then}$$

$$\mu'_1 = \varepsilon a_1 + (1-\varepsilon) a_1 \text{ and } \mu'_2 = \varepsilon b_1 + (1-\varepsilon) b_1$$

and hence $\mu'_1 > \mu'_2, \mu'_2 < \mu'_3$, and $\mu'_3 > \mu'_1$. Now for $\varepsilon$ sufficiently small $\mu'_1 \sim \mu'_1 \sim c_1$ and hence the remaining inequalities of (3.2.1) are seen to be satisfied as a consequence of the conditions satisfied by the $c_i$'s.

3.3 The next principle examined is that of "staying on a winner": Does the optimal strategy have the property that whenever a success occurs, the same random variable is required on the next trial? $S_1$, for instance, has this property. However, it is not always a characteristic of an optimal strategy, as the following example shows.

Suppose $F(p, q)$ concentrates probability $.8$ on $(.1, 0)$ and $.2$ on $(.9, 1)$. For $n = 2$, the optimum strategy is to use $X$ on the first trial and switch to $Y$ for the second only if $X$ fails on the first. Thus, the optimal strategy may not stay on a winner. Indeed, this example shows that we may stay on a loser but switch from a winner.

A property related to the intuitive notion of staying on a winner is that of "monotonicity", which we discuss for $p$ and $q$ independent. Let $S^*(n, F_G)$ denote the optimal strategy for $n$ trials against a priori $F_G$
and let \( dF^S = \frac{pdf}{\int pdf} \) and \( dF^Y = \frac{(1-p)df}{\int (1-p)df} \), with \( G^S \) and \( G^Y \) similarly defined. \( S^*(n_p, FG) \) will be termed monotone if:

(i) \( S^*(n_pF^0G) \) allows \( X \) first implies \( S^*(n_pF^0G) \) requires \( X \) first,
(ii) \( S^*(n_pFG^0) \) allows \( Y \) first implies \( S^*(n_pFG^0) \) requires \( Y \) first; and
(iii) \( S^*(n_pF^0G) \) allows \( Y \) first implies \( S^*(n_pFG^0) \) requires \( Y \) first.

(i), for instance, is to be thought of as: if a prior 'free' observation of \( X \) were allowed, then if we might use \( X \) on the first trial if \( X \) failed on the prior trial, we should certainly use \( X \) if the prior trial resulted in a success with \( X \).

Lemma 3.2. If for \( 1 \leq k \leq n-1 \) and for all \( F \) and \( G \), \( S^*(n_pFG) \) is monotone, then \( S^*(n_pFG) \) stays on a winner.

Proof. Suppose \( S^*(n_pFG) \) allows \( X \) first but does not stay on a winner for the second trial, i.e., \( S^*(n_pF^0G) \) requires \( Y \) first. Then \( S^*(n_pF^0G) \) requires \( Y \) also and hence an optimal strategy is \( Y \) followed by \( Y \) followed by an optimal continuation.

Since in such case the order of the first two trials does not effect the expected winnings, it is also optimal to use \( Y \) followed always by \( X \) and then optimal. In particular then, \( S^*(n_pF^0G) \) allows \( X \) first and hence, by monotonicity, \( S^*(n_pF^0G) \) requires \( X \) first. This implies that if \( X \) were used first and won, then \( X \) would be required next, contradicting our assumption that \( S^*(n_pFG) \) does not stay on a winner. This establishes the lemma for the second trial, but the same argument applies at any step.

With the aid of the results of Lemma 3.1 it can be verified directly that for \( n=1 \) and \( n=2 \), the optimal strategy is monotone and, therefore, for \( n=3 \) and \( p \) and \( q \) independent, the optimal strategy stays on a winner.
The general monotonicity property for independent parameters remains an open question.

In using any strategy, let \( Z_r = 1 \) if the random variable used on the \( r \)th trial wins and \( Z_r = 0 \) otherwise; i.e., \( Z_r \) is the contribution of the \( r \)th trial. The last property considered is whether, for \( S^* \) denoting the optimal \( S \), \( E[Z_r | S^*] \) is monotone increasing in \( r \). We show first that \( E[Z_r | S_1] \) is non-decreasing in \( r \) and second that \( E[Z_r | S^*] \) may decrease.

**Lemma 3.3.** \( E[Z_r | S_1] \) is non-decreasing in \( r \) for every initial distribution \( F \).

**Proof.** It is enough to prove the result for the first two trials. Suppose \( Z_1 = X \), then \( E[Z_1 | S_1] = \int \int p dF \). But \( E[Z_2 | S_1] = E \left[ E[Z_2 | Z_1, S_1] \right] = E[Z_1 | S_1] \).

In contrast to this result, consider the case of \( n=3 \), \( F(p) = \frac{5}{p^5(1-p)^3} \), and \( O(q) = q \). For optimal return, \( X \) should be employed first with expected return from the first trial of \( .9 \). If success results, then \( X \) is used again, while if failure occurs, then the criteria of Lemma 3.1 require \( Y \). The a priori expected yield from the second trial is \( \frac{64}{110} < \frac{66}{110} = .6 \).
4. The Case of One Known and One Random Probability of Success.

In this section we examine in detail the situation in which \( X \) has a binomial distribution with \( p = P_p(X=1) \) unknown but selected by a known a priori distribution, \( F \), while \( Y \) has a binomial distribution with known parameter, \( q \).

The results of the preceding section were informative largely in a negative sense; there are many nice properties which optimal strategies do not possess. Many properties which seemed obvious but which were not in general enjoyed by optimal strategies in the general case, are held by the optimal strategy when one of the random variables has a known distribution. Hence, the rather detailed proofs in this section.

We establish a series of lemmas describing some properties and the form of the optimal strategy and then obtain an explicit statement of it.

Lemma 4.1. If \( Y \) is required at any trial according to an optimal strategy, then \( Y \) is required thereafter.

Proof. First it is easily seen that if at any trial \( Y \) is required, then the optimal choice for the next trial is independent of whether \( Y \) wins or not.

Now suppose that at some trial, without loss of the first one, \( Y \) is required and is used \( r \) times but that \( X \) is allowed on the \((r + 1)^{st}\) trial. Then the expected winnings are

\[
(1) \quad rq + W_{n=r-1}(P^y, q) \int_0^1 p dF + W_{n=r-1}(P^x, q) \int_0^1 (1-p) dF
\]

\[
= rq + P_y q + W_{n=r-1}(P^x, q) \int_0^1 (1-p) dF
\]
where $F^S$ and $F^P$ are the a posteriori probabilities defined in section 3.3 and $W_n(F, q)$ is the expected winnings against a priori $F$ in $K$ trials pursuing an optimal strategy. But using $X$ first followed by $r$ trials of $Y$ yields the same amount, contradicting the fact that $Y$ was required on the first trial. Hence $Y$ must be required throughout.

As a consequence of Lemma 4.1, we can characterize an optimal strategy.

We use the notation $\mu^1 = \int_0^1 p^3 dF_p$.

**Lemma 4.2.** There exists a function, $Q_n$, of $n$ and $F$ such that for $n$ trials remaining and $F$ the a priori distribution of $p$ at that time, $Y$ is required if and only if $q \geq Q(n, F)$.

**Proof.** From Lemma 4.1, $Y$ is required if and only if

$$1_{n} q > \mu^1 + \mu^1 \sum_{n=1}^{n}(F^S_n) + (1 - \mu^1) \sum_{n=1}^{n}(F^P_n) = n K_n(F, q).$$

Now $W_n(F, q) = \max \{k, \mu_n\}$ and hence is non-decreasing, convex in $q$ for all $F$. By easy induction, $W_n(F, q)$ is non-decreasing convex in $q$ for all $F$ and $n$ and hence so is $X_n(F, q)$. Since $K_n(F, 0) = \mu_n > 0$ and $K_n(F, 1) = 1 - \frac{1 - \mu_n}{n} < 1$, it follows that for each $n$ and $F$ there is a point $Q(n, F)$ such that $q \geq Q(n, F)$ if and only if $q > K_n(F, q)$, i.e.,

if and only if $Y$ is required.

We shall adopt the convention that if $q = Q(n, F)$ we shall always use $X$, giving us a definite optimal strategy:

- If $q > Q(n, F)$, use $Y$ for all $n$ trials. If $q < Q(n, F)$, use $X$ on the first trial and compute the a posteriori distribution of $p$, $F^P$, and
compare \( q \) and \( Q(n-1, F^t) \) following the above rules for choice at the
second trial, etc.

Having characterized the optimal strategy, we turn to a series of lemmas
describing more precisely its form and properties.

**Lemma 4.3.** For all \( F \) and \( n \geq 2 \), \( Q(n, F) \geq Q(n-1, F) \).

**Proof:** Suppose the contrary. Then for \( Q(n, F) < q < Q(n-1, F) \), \( Y \) would be
required on the first trial and \( X \) on the second, contradicting Lemma 4.1.

**Lemma 4.4.** For all \( F, q, \) and \( n, \)

\[
W_n(F^s, q) \geq W_n(F, q) \geq W_n(F^s, q).
\]

**Proof:** I: \( q \geq \max \left\{ Q(n, F^s), Q(n, F), Q(n, F^s) \right\} \).

Then \( nq = W_n(F^s, q) = W_n(F, q) = W_n(F^s, q) \).

II: \( q \leq \min \left\{ Q(n, F^s), Q(n, F), Q(n, F^s) \right\} \).

We proceed by induction. The lemma holds for \( n=1 \) since for all \( q \) and \( F, \)

\[
(1) \quad \max \left\{ q, \frac{\mu_2}{\mu_1} \right\} \geq \max \left\{ q, \mu_1 \right\} \geq \max \left\{ q, \frac{1 - \mu_1}{1 - \mu_2} \right\}.
\]

In the case under consideration,

\[
(2) \quad W_n(F^s, q) = \frac{\mu_2}{\mu_1} + \frac{\mu_2}{\mu_1} W_{n-1}(F^s, q) + (1 - \frac{\mu_2}{\mu_1}) W_{n-1}(F^s, q) = \frac{\mu_2}{\mu_1} + A_n
\]

\[
(3) \quad W_n(F, q) = \frac{\mu_1}{\mu_1} + \frac{\mu_1}{\mu_1} W_{n-1}(F, q) + (1 - \frac{\mu_1}{\mu_1}) W_{n-1}(F, q) = \frac{\mu_1}{\mu_1} + B_n
\]

\[
(4) \quad W_n(F^s, q) = \frac{\mu_1 - \mu_2}{1 - \mu_1} + \frac{\mu_1 - \mu_2}{1 - \mu_1} W_{n-1}(F^s, q) + \frac{1 - \mu_1}{1 - \mu_2} W_{n-1}(F^s, q) + \frac{1 - \mu_1}{1 - \mu_2} C_n
\]

By the induction hypothesis,

\[
(5) \quad W_{n-1}(F^s, q) \geq W_{n-1}(F, q) \geq W_{n-1}(F^s, q) = W_{n-1}(F^s, q) \geq W_{n-1}(F, q) \geq W_{n-1}(F^s, q).
\]
and it is easily shown that since \( \frac{\mu_2}{\mu_1} > \mu_1 > \frac{\mu_1}{\mu_2} \), \( A_n > B_n > C_n \). 

Thus the lemma is established for this case.

As a consequence of case II, \( Q(n, F^c) \leq \min \{ Q(n, F), Q(n, F^g) \} \). For if (say) \( Q(n, F) = \min \{ Q(n, F), Q(n, F^g) \} < Q(n, F^c) \), then for \( q = Q(n, F) \), \( nq = W_n(F, q) < W_n(F^c, q) \), a contradiction of the case just established.

III: \( Q(n, F^c) < q \leq \min \{ Q(n, F), Q(n, F^g) \} \).

Then \( W_n(F^c, q) = nq \leq \min \{ W_n(F, q), W_n(F^g, q) \} \). But by an induction argument parallel that for case II it is shown that \( W_n(F, q) \leq W_n(F^g, q) \).

From case III it follows by the same reasoning as above that \( Q(n, F) \leq Q(n, F^g) \). Hence, there is only one remaining case.

IV: \( Q(n, F^c) \leq Q(n, F) < q < Q(n, F^g) \).

Immediately, \( W_n(F^c, q) = W_n(F, q) = nq \leq W_n(F^g, q) \) and the lemma is established.

Interspersed in the proof just completed is the proof of

Lemma 4.5. \( Q(n, F^c) \leq Q(n, F) \leq Q(n, F^g) \).

Lemma 4.6. Following the optimal strategy, if a success occurs on any trial, then the same random variable is used on the next trial, i.e., stay with a winner.

Proof. In view of Lemma 4.1 we need only show that if \( X \) is required and wins, then \( X \) is required on the next trial. It is clearly sufficient to show this for the first trial. Suppose to the contrary that \( Q(n, F) > q > Q(n-1, F^g) \).

By Lemma 4.3, \( Q(n-1, F^g) > Q(n-2, F^g) \geq \ldots \geq Q(1, F^g) \) and clearly

\( Q(1, F^g) > \frac{\mu_2}{\mu_1} > \mu_1 \). Hence, \( q > \mu_1 \).

By Lemma 4.5, \( q > Q(n-1, F^c) \) also. Consequently \( Y \) is required on the
second trial regardless of the outcome of first. Then

\[(1) \quad nq \leq i_n^F(q, q) = \lambda_1 + (n-1)q.\]

Hence, \(q \leq \lambda_1\) and we have a contradiction.

**Lemma 4.7** The *a priori* expected value of the yield on the \(r\)th step is non-decreasing in \(r\) when using an optimal strategy.

**Proof.** The proof can be obtained with the aid of the foregoing lemmas; it is left as an exercise for the reader.

As we have noted in Section 3, Lemma 4.7 is not in general true, while Lemma 4.6 is the "Stay on a winner" rule which, appealing as it is, does not hold in general.

With the above lemmas we are in a position to determine explicitly the value of \(Q(n, F)\). Assume that \(q = Q(n, F)\); then the optimal strategy has the following form for appropriate \(K_1^*\):

\[(4.1) \quad (A) \text{ Observe } X \text{ until a failure occurs.} \]

\[(B) \text{ There exists an integer } K_1 \geq 0 \text{ such that if at least } K_1 \text{ successes preceded the first failure, continue with } X \text{ and otherwise switch to } Y \text{ for the remaining trials.} \]

\[(C) \text{ There is an integer } K_2 \geq 0 \text{ attached to the second failure such that if at least } K_1 + K_2 \text{ successes with } X \text{ precede the second failure of } X, \text{ continue with } X \text{ and otherwise switch to } Y \text{ for the remaining trials.} \]

\[(D) \text{ In general, let } S_r \text{ be the number of successes that precede the } r\text{th failure of } X. \text{ If } S_r \geq K_1 + K_2 + \cdots + K_r, \text{ continue with } X \text{ and otherwise switch to } Y \text{ for the remaining trial.} \]
Thus, any sequence \( K = (K_1, K_2, \ldots, K_n) \) of integers, \( 0 \leq K_\lambda \leq n \), corresponds to a strategy of the same form as the optimal.

Let \( E_k \) denote expectation given \( K \) and \( P \) and \( E_k \) denote expected value given \( K \) and \( p \).\(^1\) In using any strategy for \( n \) trials, \( X \) will be used a certain number, \( M_x \), of times and there will be a certain number, \( S_x \), of successes with \( X \); similarly for \( Y \).

**Theorem 4.1.** \( Q(n_2, P) = \max_k \left\{ \frac{E_k[S_x]}{E_k[N_x]} \right\} \).

**Proof.** \( q = Q(n_2, P) \) implies \( nq = W_n(P, q) \). But since the optimal strategy corresponds to a sequence \( K \), this is equivalent to \( nq = \max_k \left\{ E_k[S_x] + E_k[y] \right\} \).

However, \( E_k[S_x] = qE_k[N_y] \) and neither \( E_k[S_x] \) nor \( E_k[N_y] \) depends on \( q \). Hence \( q = Q(n_2, P) \) implies each of the following equivalent statements:

\[
\begin{align*}
q &= \max_k \left\{ E_k[S_x] + q E_k[N_y] \right\} \\
q &\geq E_k[S_x] + q E_k[N_y] \quad \text{for all } K \text{ with equality for some } q(n - E_k[N_q]) \geq E_k[S_x] \\
q &\geq \frac{E_k[S_x]}{E_k[N_x]} \\
q &= \max_k \left\{ \frac{E_k[S_x]}{E_k[N_x]} \right\}.
\end{align*}
\]

\(^1\) The authors are indebted to the referee for suggesting the following derivation of \( Q(n_2, P) \) which is somewhat simpler and more illuminating than that originally used to obtain the result.
Corollary.

\[ Q(n, F) = \max_k \left\{ \left( \int_0^1 \frac{E_{\text{kp}} \{ N_x \} \text{dF}}{\sum \text{E}_{\text{kp}} \{ N_x \} \text{dF}} \right)^k \right\} = \max_k \left\{ \left( \int_0^1 \frac{(n-E_{\text{kp}} \{ N_y \}) \text{dF}}{\sum (n-E_{\text{kp}} \{ N_y \}) \text{dF}} \right)^k \right\} \]

We give two methods of evaluating \( Q(n, F) \). The first proceeds by obtaining directly a formula for \( E_{\text{kp}} \{ N_y \} \) and yields

\[ \sum_{j=1}^{\beta(j)-1} \left( \frac{\beta(j)}{\sum_{j=1}^{\beta(j)-1} f_i} \right)^{K_i+1} \]

where \( \beta(j) = \max \{ i : \sum_{j=1}^{i} k_j + i < j \} \) = number of failures of \( X \) in the first \( (j-1) \) trials and \( \sum_{j=1}^{\beta(j)-1} f_i \) denotes the sum over all choices of \( f_i \)'s such that \( f_i \geq 0 \), \( f_i = 0 \), \( f_o = 0 \), if \( K_{i+1} = 0 \), \( \sum_{i=0}^{v} f_i \leq v \) and

\[ \sum_{i=1}^{\beta(j)} f_i = \beta(j) \] (\( f_i \) denotes the number of failures between the \( (K_1 + K_2 + \cdots + K_i) \) th and \( (K_1 + K_2 + \cdots + K_{i+1}) \) th success).

The second proceeds by obtaining directly a formula for \( E_{\text{kp}} \{ S \} = E_{\text{kp}} \{ N_x \} \).

While more complicated in appearance and derivation, it is the result of a direct counting.
\[(4.3) \quad E_{kp}[S_x] = \sum_{r=0}^{\infty} I_r \rho \quad \text{where} \]

\[
I_r = \sum_{a_1=0}^{1} \frac{a_1^{-1}}{(a_1^{-1})^\ldots(a_1^{-1})} \cdot \ldots \cdot \frac{a_1^{-1}}{(a_1^{-1})^\ldots(a_1^{-1})} \sum_{a_2=1}^{2-a_1-1} \left( \frac{a_2^{-1}}{(a_2^{-1})^\ldots(a_2^{-1})} \right) \ldots \sum_{a_r=1}^{r-a_1-\ldots-a_{r-1}-1} \left( \frac{a_r^{-1}}{(a_r^{-1})^\ldots(a_r^{-1})} \right) \]  

\[
\frac{b_r}{b_r} \cdot \frac{b_r}{b_r} \cdot \ldots \cdot \frac{b_r}{b_r} = \prod_{i=1}^{r} b_i^{-1} = r^{-1} 
\]

\[
n = \sum_{i=1}^{r} k_i \cdot \rho^n 
\]

\[
\sum_{r=1}^{k_1 \cdot k_2 \cdot \ldots \cdot k_r} \rho^{(1-p)r} 
\]

with \( b_r = r = a_1 \cdot a_2 \cdot \ldots \cdot a_r \) and we interpret \( \binom{-1}{1} = 1 \).

Some special cases are worth noting:

\[
Q(2, F) = \frac{1}{\int_0^{1} (1+p) \, dF} \int_0^{1} (p+p^2) \, dF
\]

and

\[
Q(3, F) = \max \left( \frac{1}{\int_0^{1} (1+p) \, dF} \int_0^{1} (p+p^2+p^3) \, dF, \frac{1}{\int_0^{1} (1+p) \, dF} \int_0^{1} (p+2p) \, dF \right)
\]

Each term of \( Q(3, F) \) can occur, e.g., for \( F(p) = p \) the first is the maximum (value \( \frac{13}{22} \)), while for \( F(p) = p^{1/5} \) the second is the maximum (value \( \frac{23}{88} \)).

The expression for \( Q(n, F) \) cannot be simplified in any essential way, which again testifies to the complex nature of the optimal strategies in sequential design problems.
If one chooses $K_1 = r$, $K_2 = K_3 = \ldots = 0$, then

$$\frac{E[S_1]}{E[N]} = \frac{\mu_1 + \mu_2 + \ldots + \mu_r + (n-r) \mu_{r+1}}{1 + \mu_2 + \ldots + \mu_r + (n-r) \mu_r}$$

where $\mu'_r = \int_0^p p^r dF(p)$. For distributions such that $\sum_{k=1}^n \mu'_k = \infty$, at least, a reasonable approximation to $Q(n_2F)$ may be had by taking $r = n-1$ and using

$$L(n_2F) = \frac{\mu'_1 + \mu'_2 + \ldots + \mu'_n}{1 + \mu'_1 + \ldots + \mu'_{n-1}}$$

in place of $\mathcal{D}(n_2F)$. For the uniform distribution, $Q$ and $L$ coincide for $n \leq n_0$ but not for larger $n$. It is worth noting that $L$ shares many of the properties of $Q$.

Lemma 4.6. $L$ is non-decreasing in $n$ and $L(n_2F^S) \geq L(n_2F) \geq L(n_2F^F)$.

Proof: The proof is based on the well known result that for $a_r > 0$ and $b_r > 0$, \( \left\{ \frac{a_r}{b_r} \right\} \) an increasing sequence implies \( \left\{ \frac{a_1 + a_2 + \ldots + a_n}{b_1 + b_2 + \ldots + b_n} \right\} \) is an increasing sequence. Since

$$\int_0^1 p^{r-1} dF \leq \int_0^1 p^{r-1} dF$$

by virtue of Holder's inequality, we have $L(n_2F)$ increasing in $n$. The second conclusion can be readily proved using these same results.

We close by noting that if the number of trials is sufficiently large one almost always should commence by using $X$. More precisely we see from Eq (4.14) that as $n \to \infty$, $\mathcal{D}(n_2F)$ becomes at least $\frac{\mu_{r+1}}{\mu_r}$, and this for every $r$. But $\frac{\mu_{r+1}}{\mu_r}$ is the expected value of $p$ given $r$ successes
which will tend to the supremum of the spectrum of \( F \) as \( r \) increases. Clearly, if \( q \) is greater than the supremum of the spectrum of \( F \), one would never play \( X \), while if \( q \) is less than the supremum, for all sufficiently large \( n \), \( Y \) should be used first. For a fixed \( n \) we have the following.

**Lemma 4.9.** Given \( n \) and \( F \), \( Y \) should never be used if

\[
q < \frac{\int_0^1 p(1-p)^{n-1} \, dp}{\int_0^1 (1-p)^{n-1} \, dp}
\]

**Proof.**

If \( q < E[p \mid n-1 \text{ failures}] \), then \( q < Q(1, F^{n-1}) \) where \( F_1^{i_1} \ldots^{j_1} \) denotes the a posteriori distribution of \( p \) after \( i \) failures and \( j \) successes of \( X \). But, by repeated application of lemmas 4.3 and 4.5,

\[
Q(1, F^{i_1} \ldots^{j_1}) < Q(2, F^{i_1} \ldots^{j_1}) < Q(2, F^{i_2} \ldots^{j_2}) < Q(2, F^{i_3} \ldots^{j_3})
\]

for all \( i + j = n-2 \). By repetition of this argument we obtain that for all \( r \leq n \),

\[
q < Q(1, F^{i_1} \ldots^{j_1}) < Q(r, F^{i_1} \ldots^{j_1})
\]

for all \( i + j = n - r \) and hence \( Y \) is never used in the sequence of \( n \) trials.

5. **An Applied Problem.**

An interesting problem in industrial inspection is closely related to the problem of Section 4. Suppose that lots of \( n \) items are produced by a process having probability \( p \) of producing a defective where \( p \) varies
from lot to lot according to an a priori distribution, $F(p)$. Let the loss per defective item accepted be unity and the cost of inspection be $c$ per item inspected ($c < 1$). Items are drawn and inspected (defective items found being replaced by good items at no additional cost) until a sequential stopping rule terminates inspection at which point the remainder of the lot is accepted. A stopping rule is desired which will minimize the expected loss.

One may proceed to attack this problem in the spirit of Section 4 and find a completely analogous series of lemmas culminating in the theorem that for $n$ items remaining and a priori distribution $F$, it is optimal to inspect another item or to accept the remaining $n$ according as $c$ is less than or greater than $Q(n, F)$. The same result is more immediately obtained by noting that the problem is equivalent to finding a rule to maximize the gain if one wins $c$ for each item not inspected, nothing for each good item inspected, and one for each defective item inspected (and replaced). This latter problem is precisely that treated in Section 4.
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Additional copies for project leaders and assistants and reserve for future requirements