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Adapting to Unknown Noise Distribution in Matrix Denoising

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Abstract

We consider the problem of estimating an unknown matrix $X \in \mathbb{R}^{m \times n}$, from observations $Y = X + W$ where $W$ is a noise matrix with independent and identically distributed entries, as to minimize estimation error measured in operator norm. Assuming that the underlying signal $X$ is low-rank and incoherent with respect to the canonical basis, we prove that minimax risk is equivalent to $(\sqrt{m} + \sqrt{n})/\sqrt{I_W}$ in the high-dimensional limit $m,n \to \infty$, where $I_W$ is the Fisher information of the noise. Crucially, we develop an efficient procedure that achieves this risk, adaptively over the noise distribution (under certain regularity assumptions).

Letting $X = U \Sigma V^T$—where $U \in \mathbb{R}^{m \times r}$, $V \in \mathbb{R}^{n \times r}$ are orthogonal, and $r$ is kept fixed as $m,n \to \infty$—we use our method to estimate $U, V$. Standard spectral methods provide non-trivial estimates of the factors $U, V$ (weak recovery) only if the singular values of $X$ are larger than $(mn)^{1/4} \text{Var}(W_{11})^{1/2}$. We prove that the new approach achieves weak recovery down to the information-theoretically optimal threshold $(mn)^{1/4} I_W^{1/2}$.

1 Introduction and main result

Let $X \in \mathbb{R}^{m \times n}$ be an unknown signal, and assume we are given observations of its entries corrupted by additive noise:

$$Y = X + W,$$

where $W = (W_{ij})_{i \leq m, j \leq n}$ has i.i.d. entries $W_{ij} \sim p_W$. We would like to estimate $X$ in operator norm, namely construct an estimator $\hat{X} : \mathbb{R}^{m \times n} \to \mathbb{R}^{m \times n}$ as to minimize $\mathbb{E}\{ \| \hat{X}(Y) - X \|_{op} \}$. Random matrix theory [AGZ09, BS10a] characterizes the unbiased estimator $\hat{X}_{ub}(Y) = Y$ as $m,n \to \infty$ (under suitable tail conditions on the law of the noise):

$$\mathbb{E}\{ \| \hat{X}_{ub}(Y) - X \|_{op} \} = (\sqrt{m} + \sqrt{n})\sqrt{\text{Var}(W_{11})} + O(1).$$

Anderson’s lemma [And55a] implies that the unbiased estimator is minimax optimal when the noise is Gaussian. However, for more general noise distributions, the result (1.2) leads to a natural statistical question:

*Can we improve over the error of the unbiased estimator?*
This question was answered positively in several cases, under the assumption that \( X \) is low rank and the noise distribution \( p_W \) is known [DM15, LKZ15, PWBM16]. It is quite easy to understand a mathematical mechanism leading to this answer [LKZ15, PWBM16] (a somewhat different mechanism is studied in [DM15]). In many cases of interest (e.g. when \( X = uv^T \) with \( u \in \mathbb{R}^m, v \in \mathbb{R}^n \) incoherent with respect to the canonical basis), the entries of \( X \) are much smaller than the entries of the noise matrix \( W \). Imagine to apply a non-linear denoiser \( f : \mathbb{R} \to \mathbb{R} \) component-wise to \( X \) to obtain \( \hat{X}(Y) = f(Y) \). By Taylor expansion we have

\[
\hat{X}(Y) \approx f(W) + f'(W) \odot X \approx \mathbb{E}\{f'(W)\}X + f(W),
\]

where \( \odot \) denotes Hadamard (entry-wise) product, and \( W \sim p_W \) is a scalar with the same distribution as the entries of \( W \). Notice that \( f(W) \) is a matrix with i.i.d. entries. Therefore, for any nonlinearity such that \( \mathbb{E}\{f'(W)\} = 1, \mathbb{E}\{f(W)\} = 0 \), we expect \( \mathbb{E}\{\|\hat{X}(Y) - X\|_{op}\} = (\sqrt{m} + \sqrt{n})\sqrt{\mathbb{E}\{f(W)^2\}} + O(1) \). This suggests to choose the nonlinearity \( f \) by solving the optimization problem

\[
\text{minimize} \quad \mathbb{E}\{f(W)^2\}, \quad (1.4)
\]

subject to \( \mathbb{E}\{f(W)\} = 0, \mathbb{E}\{f'(W)\} = 1. \) (1.5)

Assuming \( p_W \) to have a differentiable density (also denoted by \( p_W \)), a simple application of Cauchy-Schwartz inequality implies that the optimal \( f(x) \) is given by

\[
f_W(x) = -\frac{1}{I_W} \frac{p_W'(x)}{p_W(x)}, \quad I_W = \int_{\mathbb{R}} \frac{(p_W'(x))^2}{p_W(x)} \, dx. \quad (1.6)
\]

Notice that \( I_W \) is the Fisher information for the location family \( \{p_{W}(W + \theta)\}_{\theta \in \mathbb{R}} \). This denoiser would achieve \( \mathbb{E}\{\|\hat{X}(Y) - X\|_{op}\} = (\sqrt{m} + \sqrt{n})I_W^{1/2} + O(1) \).

Unfortunately, the above argument does not provide a concrete statistical procedure, since the noise distribution \( p_W \) is typically unknown to the data analyst. Our main result is that the ideal estimation error can be achieved without prior knowledge of \( p_W \). In order to state this result, we define a class of matrices \( \mathcal{F}_{m,n}(r, M, \eta) \) that formalizes the assumption that \( X \) has small entries and is low-rank:

\[
\mathcal{F}_{m,n}(r, M, \eta) \equiv \{ X \in \mathbb{R}^{m \times n} : \text{rank}(X) \leq r, \|X\|_{op} \leq M(m \vee n)^{1/2}, \|X \odot X\|_{op} \leq M(m \vee n)^{1/2 - \eta}, \|X\|_{\ell_2 \to \ell_{\infty}} \leq Mn^{1/2 - \eta} \text{ and } \|X^T\|_{\ell_2 \to \ell_{\infty}} \leq Mr^{1/2 - \eta} \}. \quad (1.7)
\]

(Recall that \( \|X\|_{\ell_2 \to \ell_{\infty}} \) is the maximum \( \ell_2 \) norm of any row of \( X \).)

We also denote by \( \mathcal{F}_{m,n}(M, \eta) \) the same class, where the rank constraint is removed, i.e. \( \mathcal{F}_{m,n}(M, \eta) = \mathcal{F}_{m,n}(m \wedge n, M, \eta) \). Our first result concerns estimation in the class \( \mathcal{F}_{m,n}(M, \eta) \).

**Theorem 1.** For each \( m, n \in \mathbb{N} \), there exists an estimator \( \hat{X}^{(s)} : \mathbb{R}^{m \times n} \to \mathbb{R}^{m \times n} \) such that the following holds. Assume that the noise distribution is absolutely continuous with respect to Lebesgue measure, with density \( p_W \) satisfying the following assumptions:

A0. The Fisher information \( I_W \) is finite.

A1. \( \int |w|^2 p_W(w) \, dw \leq M_1 \) for some constant \( M_1 \).
A2. \( p_W \in C^3(\mathbb{R}) \), with derivatives \( p_W^{(\ell)} \) satisfying
\[
\left\| p_W^{(\ell)}(\cdot) \right\|_\infty \leq M_2 \quad \text{for all } \ell \in \{0, 1, 2\}.
\] (1.8)

Then, letting \( Y = X + W \), with \( (W_{ij})_{i \leq m, j \leq n} \sim_{i.i.d} p_W \), assuming \( m \asymp n \), we have:
\[
\sup_{X \in \mathcal{F}_{m,n}(M,\eta)} \mathbb{E}\{ \| \hat{X}(\cdot)(Y) - X \|_{op} \} \leq \frac{\sqrt{m} + \sqrt{n}}{\sqrt{\text{tr} W}} + o(m \vee n)^{1/2}.
\] (1.9)

We next consider the rank-constrained class \( \mathcal{F}_{m,n}(r, M, \eta) \).

**Theorem 2.** For each \( m, n \in \mathbb{N} \), there exists an estimator \( \hat{X} : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^{m \times n} \) such that the following holds. Under the same assumptions of Theorem 1, letting \( Y = X + W \), with \( (W_{ij})_{i \leq m, j \leq n} \sim_{i.i.d} p_W \), assuming \( m \asymp n \), we have:
\[
\sup_{X \in \mathcal{F}_{m,n}(r, M, \eta)} \mathbb{E}\{ \| \hat{X}(Y) - X \|_{op} \} \leq \frac{\sqrt{m} \vee \sqrt{n}}{\sqrt{\text{tr} W}} + o((m \vee n)^{1/2}).
\] (1.10)

**Remark** The upper bound of Eq. (1.9) improves over the risk of the unbiased estimator in Eq. (1.2) in that the factor \( \sqrt{\text{Var}(W)} \) is replaced by \( 1/\sqrt{\text{tr} W} \), which is strictly smaller in any case except for Gaussian noise (in which case the two bounds are equal). This is achieved by entry-by-entry denoising of the matrix \( Y \), following the ideas described above.

The bound of Eq. (1.10) further improves the factor \( (\sqrt{m} + \sqrt{n}) \) to \( \sqrt{m} \vee \sqrt{n} \) by exploiting the low-rank structure in \( X \). Entrywise denoising is followed by suitable shrinkage of the singular values of \( Y \). Singular values shrinkage is a well studied topic, see for instance [SN13, GD14, Cha15, DG14, DGJ18]. However, by itself, it does not achieve the minimax error for non-Gaussian noise.

Our second main result establishes that no estimator can do substantially better than what guaranteed by , in the rank constrained case, even with knowledge of \( p_W \).

**Theorem 3.** Assume the density \( p_W \) to be weakly differentiable and \( \text{tr} W \) to be finite, and assume \( m = m(n) \) be such that \( \lim_{n \rightarrow \infty} m(n)/n = \gamma \). Then
\[
\liminf_{n \rightarrow \infty} \sup_{\hat{X}(\cdot) \in \mathcal{F}_{m,n}(r, M, \eta)} \frac{1}{(mn)^{1/4}} \mathbb{E}\{ \| \hat{X}(Y) - X \|_{op} \} \geq \frac{\gamma^{1/4} \vee \gamma^{-1/4}}{\sqrt{\text{tr} W}} - o_M(1).
\] (1.11)

(Here the infimum is taken over all measurable functions \( \hat{X} : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^{m \times n} \), and \( o_M(1) \) is a quantity vanishing as \( M \rightarrow \infty \).)

**Remark** Roughly speaking, Theorem 3 states that, in the rank constrained class, any estimator has worst case error lower bounded as \( \sup_{X \in \mathcal{F}_{m,n}(r, M, \eta)} \mathbb{E}\{ \| \hat{X}(Y) - X \|_{op} \} \gtrsim (\sqrt{m} \vee \sqrt{n})/\sqrt{\text{tr} W} \). This lower bound obviously applies to the larger class \( \mathcal{F}_{m,n}(M, \eta) \), but it falls short of the upper bound of Theorem 1 by a factor \( (\sqrt{m} + \sqrt{n})/(\sqrt{m} \vee \sqrt{n}) \leq 2 \) (for large \( m, n \)).

It is an open problem to close this gap between upper and lower bounds in the rank-unconstrained class.

The rest of this paper is organized as follows. Section 2 overviews related work. Section 3 describes the construction of the adaptive estimator, and states a more detailed version of Theorem 2. Section 4 applies the adaptive denoiser to the problem of estimating the top singular subspaces of a low-rank matrix \( X \). Section 5 illustrates the performance of our method through a simulation study. Finally, Sections 6 and 7 outline the proofs of our main theorems, with details deferred to the appendices.
2 Related work

The problem of estimating a noisy matrix in operator norm is central in principal component analysis and covariance estimation [Ver12], matrix completion [CP10, KMO10], graph localization [Sin08, JM13], group synchronization [Sin11, WS13, JMRT16], community detection [Moo17, Abb18], gene expression analysis [Rin08], and a number of other applications.

A large part of the theoretical literature has focused on the Gaussian spiked model, or on sparse covariance structures. In the last setting, several estimators are known that achieve minimax error rates to varying degree of generality [EK08, BL08, CZZ10, CMW13]. These approaches are often based on entry-wise thresholding of the empirical covariance. While sparsity is a useful assumption for some datasets, it is not warranted for other applications (including matrix completion, group synchronization, localization, and community detection). Accordingly, we do not assume $X$ to be sparse, but rather incoherent with respect to the canonical basis. Our focus is on achieving adaptivity with respect to the unknown noise distribution. In order to investigate the effect of the noise distribution, we cannot limit ourselves to determine error rates, but instead we need to pinpoint the asymptotic estimation error up to sub-leading error terms.

Our work is closely related to results on the spiked covariance model. Within the spiked model, we observe $m$-dimensional vectors $(y_i)_{i\le n} \sim N(0, \Sigma)$ with covariance $\Sigma = UU^T + I_m$, with $U \in \mathbb{R}^{m \times r}$ a matrix describing the $r \ll m$ spikes. If $y_1, \ldots, y_m$ are stacked as columns of a matrix $Y$, we can equivalently write $Y = X + W$, where $X = UV^T$, $V \in \mathbb{R}^{n \times r}$ has entries $(V_{ij})_{i\le n, j\le r} \sim \text{iid } N(0, 1/m)$, and $(W_{ij})_{i\le m, j\le n} \sim N(0, 1)$. The low-rank component $UU^T$ of the covariance is extracted using principal component analysis, which is known to be optimal, but is not asymptotically consistent in the high-dimensional regime $n \approx m$ unless $\sigma_{\min}(U) \to \infty$ [JL09]. Random matrix theory has been used to determine asymptotic detection (or weak recovery) thresholds, asymptotic estimation errors, as well optimal hypothesis testing procedures [BAP05, BS06, Pau07, OMH13].

A substantial literature studies matrix denoising under a Gaussian noise model [SN13, GD14, Cha15, DG14, DGJ18]. By rotational invariance, in this case optimal denoising is achieved by a form of singular value shrinkage, i.e. taking the singular value decomposition of the observed matrix $Y$ and applying a nonlinear function to its singular values. Singular value shrinkage comes with error guarantees also under non-Gaussian noise models (among others, [Cha15] studies a broad class of models). However, it is suboptimal in this more general case.

Singular value shrinkage is also suboptimal when the signal $X = UV^T$ is of rank $r \ll m, n$, and the factors $U, V$ have additional structure. Sparse principal component analysis is a prominent example of this phenomenon [JL09, ZHT06, dGJL05]. More recently, the case in which the rows of $U, V$ are i.i.d. draws from probability distributions $p_U, p_V$ on $\mathbb{R}^r$ has attracted attention, see e.g. [LKZ17, DM14, DAM16, KZX16, BDM+16, LM16, Mio17, MV17].

As suggested by the simple argument in the introduction, singular value shrinkage (and other simple spectral methods) also becomes suboptimal when the noise is non-Gaussian. Namely, by applying a suitable entry-wise nonlinearity to the data matrix $Y$, we can reduce the operator norm of the noise, without affecting the signal. This phenomenon was investigated in [DM15] in the context of the hidden sub-matrix problem, and in [KXZ16] in the context of rank-one matrix estimation for symmetric matrices. In particular, [KXZ16] assumed the rank-one signal $X = \sqrt{n}uu^T$ to be incoherent with the standard basis and identified a weak recovery phase transition when the Fisher information crosses $I_W = 1$ (assuming the normalization $\|u\|_2 = 1$). Subsequently, Perry and co-authors [PWBM16] proved that the same threshold correspond to a phase transition in hypothesis
testing. If \( I_W < 1 \) it is impossible to distinguish with vanishing error probability between data
from the spiked model \( Y = \sqrt{n}uu^T + W \), and pure noise \( Y = W \). If \( I_W > 1 \), instead the two
hypotheses can be distinguished with vanishing error probability as \( n \to \infty \).

3 Construction of the adaptive estimators

In this section we describe the estimators \( \hat{X}(Y) \) and \( \hat{X}(Y) \) of Theorems 1 and 2.

Equation (1.6) suggest to estimate the density of the noise \( p_W(\cdot) \), in order to construct an
approximation of the denoiser \( f_W(\cdot) \). The challenge is of course that we do not have samples from
\( p_W \) but only matrix entries \( (Y_{ij})_{i \leq m,j \leq n} \). However, it turns out that we can use these as surrogates
for the noise values \( (W_{ij})_{i \leq m,j \leq n} \), provided we introduce a suitable noise correction.

Let \( K : \mathbb{R} \to \mathbb{R} \) be a first order non-negative kernel, i.e,

\[
K(z) \geq 0, \quad \int_{\mathbb{R}} K(z) \, dz = 1 \quad \text{and} \quad \int_{\mathbb{R}} zK(z) \, dz = 0.
\] (3.1)

Throughout, we assume \( K(\cdot) \) to satisfy the following smoothness condition. For some constant
\( M > 0 \) (independent of the model parameters \( X, p_W, m, n \)) and for \( \ell \in \{0, \ldots, 3\} \), we have

\[
\left\| K^{(\ell)} \right\|_{\infty} \leq M, \quad \int_{\mathbb{R}} \left( K^{(\ell)}(z) \right)^2 \, dz \leq M, \quad \int_{\mathbb{R}} z^2 K(z) \, dz \leq M.
\] (3.2)

Many standard kernels satisfy these conditions, a simple example being the Gaussian kernel \( K(z) = \frac{1}{\sqrt{2\pi}} \exp(-z^2/2) \).

We further define the function

\[
H(\sigma) = \left\{ \begin{array}{ll}
\sqrt{(\sigma + \gamma^{-1/2}\sigma^{-1})(\sigma + \gamma^{1/2}\sigma^{-1})} & \text{if } \sigma \geq 1, \\
\gamma^{1/4} + \gamma^{-1/4} & \text{otherwise.}
\end{array} \right.
\] (3.3)

Note that \( \sigma \mapsto H(\sigma) \) is strictly increasing on \((1, \infty)\), and hence we can define its inverse \( H^{-1}(y) \) on \((H(1), \infty)\) with \( H(1) = \gamma^{1/4} + \gamma^{-1/4} \).

Our adaptive estimators depends on parameters \( h_n, h'_n, \varepsilon, \delta > 0 \) and proceeds as follows:

1. Compute the average entry \( \bar{Y} \equiv \sum_{i \leq m, j \leq n} Y_{ij} / (mn) \).

2. Compute the corrected kernel estimates

\[
\hat{p}_Y(x) = \frac{1}{mn h_n} \sum_{i \leq m, j \leq n} K\left( \frac{Y_{ij} - \bar{Y} - x}{h_n} \right), \quad \hat{p}'_Y(x) = \frac{1}{mn(h'_n)^2} \sum_{i \leq m, j \leq n} K'\left( \frac{Y_{ij} - \bar{Y} - x}{h'_n} \right).
\] (3.4)

3. Define the denoiser \( \hat{f}_{Y,\varepsilon}(\cdot) \), and the estimated Fisher information

\[
\hat{f}_{Y,\varepsilon}(x) = -\frac{\hat{p}'_Y(x)}{\hat{p}_Y(x)} + \varepsilon, \quad \hat{I}_{W,\varepsilon} = \frac{1}{mn} \sum_{i \leq m, j \leq n} \left( \frac{\hat{p}'_Y(Y_{ij} - \bar{Y})}{\hat{p}_Y(Y_{ij} - \bar{Y}) + \varepsilon} \right)^2 + \varepsilon.
\] (3.5)

(We will omit the dependence on \( \varepsilon \) whenever clear from the context.)

5
4. Construct the matrix obtained by applying this denoiser entry-wise to \( Y \): 
\[
\hat{X}^{(0)}(Y; h_n, h'_n, \varepsilon) = \hat{f}_Y(Y),
\]
and compute its singular value decomposition 
\[
\hat{X}^{(0)} = (mn)^{1/4} \hat{U} \hat{\Sigma}^{(0)} \hat{V}^T.
\] (3.6)

5. Construct the diagonal matrix \( \hat{\Sigma} \) with entries 
\[
\hat{\Sigma}_{i,i} = \begin{cases} 
\hat{I}^{-1/2} W^{-1} \hat{\Sigma}^{(0)}_{i,i} H^{-1}(1 + \delta)^{1/2} \hat{I}^{-1/2} W^{-1} \hat{\Sigma}^{(0)}_{i,i} & \text{if } \hat{\Sigma}^{(0)}_{i,i} \geq (1 + \delta) H^{1/2} \hat{I}^{-1/2} W^{-1} \hat{\Sigma}^{(0)}_{i,i} \\
0 & \text{otherwise}.
\end{cases}
\] (3.7)

6. Return 
\[
\hat{X}((Y; h_n, h'_n, \varepsilon, \delta)) = \hat{I}^{-1} W^{-1} \hat{X}^{(0)}(Y; h_n, h'_n, \varepsilon),
\] (3.8)
\[
\hat{X}(Y; h_n, h'_n, \varepsilon, \delta) = (mn)^{1/4} \hat{U} \hat{\Sigma} \hat{V}^T.
\] (3.9)

(In what following we will often omit the dependence on the parameters \( h_n, h'_n, \varepsilon, \delta \).)

**Remark** Note that \( \hat{p}_Y(x), \hat{p}'_Y(x) \) are estimators for the noise density \( p_W \) and its first derivative. They differ from standard kernel estimators because of the shift \( Y \). This is designed to correct the error made because \( Y_{ij} \neq W_{ij} \). If all the entries of \( X \) were equal to \( X \), we would have \( W_{ij} = Y_{ij} - X \). The correction we use simply replaces \( X \) by \( Y \). It turns out that this simple correction is sufficient in the large \( m, n \) limit.

The next statement provides a more detailed version of Theorems 1.

**Theorem 4.** Assume that conditions A0, A1, A2 of Theorem 1 hold. Further, let \( h_n = n^{-\eta_1} \) and \( h'_n = n^{-\eta_2} \) for some \( \eta_1 \in (1/4, 1) \) and \( \eta_2 \in (1/4, 1/3) \). Then the estimators \( \hat{X}((Y; h_n, h'_n, \varepsilon)) \) and \( \hat{X}(Y; h_n, h'_n, \varepsilon, \delta) \) defined above satisfies 
\[
\limsup_{n \to \infty} \sup_{X \in F_m,n(r,M,n)} \frac{1}{(mn)^{1/4}} \mathbb{E} \left\| \hat{X}((Y; h_n, h'_n, \varepsilon)) - X \right\|_{op} \leq (\gamma^{1/4} + \gamma^{-1/4}) I_W^{-1/2} + o_{\varepsilon, \delta}(1),
\] (3.10)
\[
\limsup_{n \to \infty} \sup_{X \in F_m,n(r,M,n)} \frac{1}{(mn)^{1/4}} \mathbb{E} \left\| \hat{X}(Y; h_n, h'_n, \varepsilon, \delta) - X \right\|_{op} \leq \max\{\gamma^{1/4}, \gamma^{-1/4}\} I_W^{-1/2} + o_{\varepsilon, \delta}(1),
\] (3.11)

where \( \lim_{\delta \to 0} \lim_{\varepsilon \to 0} o_{\varepsilon, \delta}(1) = 0 \).

4 **Singular space recovery and additional results**

An important application of the denoising method introduced above is to estimate latent factors in high-dimensional data. We denote the singular value decomposition of the signal \( X \) by 
\[
X = (mn)^{1/4} U \Sigma V^T,
\] (4.1)
where \( \mathbf{U} \in \mathbb{R}^{m \times r}, \mathbf{V} \in \mathbb{R}^{n \times r} \) are orthogonal matrices (i.e. satisfying \( \mathbf{U}^T \mathbf{U} = \mathbf{V}^T \mathbf{V} = \mathbf{I}_r \)), and \( \Sigma \in \mathbb{R}^{r \times r} \) is a diagonal matrix containing the scaled singular values of \( \mathbf{X} \) (denoted by \( \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r \)). The scaling factor \((mn)^{1/4}\) is introduced so that the operator norm of \( \mathbf{X} \) is of the same order as the one of \( \mathbf{W} \) when the \( \sigma_i \)'s are of order one.

In the following, we denote by \( \mathbf{U}_k \in \mathbb{R}^{m \times k} \) the matrix containing the first \( k \) columns of \( \mathbf{U} \), by \( \mathbf{V}_k \in \mathbb{R}^{n \times k} \) the matrix containing the first \( k \) columns of \( \mathbf{V} \), and so on. We also denote by \( u_i \) the \( i \)-th column of \( \mathbf{U} \), and by \( v_i \) the \( i \)-th column of \( \mathbf{V} \).

Our next theorem bounds the principal angle between the eigenspaces \( \mathbf{U}_k, \mathbf{V}_k \), and their estimates \( \hat{\mathbf{U}}_k, \hat{\mathbf{V}}_k \). Define the function

\[
G(\sigma; t) \equiv \left( \frac{1 - t^{-2}\sigma^{-4}}{1 + (\gamma^{1/2} \wedge \gamma^{-1/2})t^{-1}\sigma^{-2}} \right)^{1/2}.
\]

**Theorem 5.** Assume that conditions A0, A1, A2 of Theorem 1 hold. Further, let \( h_n = n^{-\eta_1} \) and \( h'_n = n^{-\eta_2} \) for some \( \eta_1 \in (1/4, 1) \) and \( \eta_2 \in (1/4, 1/3) \). Consider a sequence of matrices \( \mathbf{X} = \mathbf{X}_n \in \mathcal{F}_{m,n}(r,M,\eta) \) as in Eq. (4.1) with scaled singular values \( \Sigma \) independent of \( n \). For any \( l \in [r] \) such that \( \sigma_l \neq \sigma_{l+1} \) and \( \sigma_l > \mathbf{I}_W^{-1/2} \), the following holds

\[
\lim_{n \to \infty, m/n \to \gamma} \sigma_{\min}\left( \hat{\mathbf{U}}_l^T \mathbf{U}_l \right) = G(\sigma_l; \mathbf{I}_W) + o_{\varepsilon,\delta}(1) \tag{4.2}
\]

where \( \lim_{\delta \to 0} \lim_{\varepsilon \to 0} o_{\varepsilon,\delta}(1) = 0 \).

Note that this theorem implies the following phase transition phenomenon.

**Corollary 4.1.** Under the assumptions of Theorem 5

\[
\begin{align*}
\sigma_{\min}^2\left( \hat{\mathbf{U}}_k^T \mathbf{U}_k \right) &= o_P(1), & \text{if } \sigma_k < 1/\sqrt{\mathbf{I}_W}, \\
\sigma_{\min}^2\left( \hat{\mathbf{U}}_k^T \mathbf{U}_k \right) &\geq \varepsilon_0 + o_P(1), & \text{if } \sigma_k > 1/\sqrt{\mathbf{I}_W}, \sigma_k > \sigma_{k+1},
\end{align*}
\]

where \( \varepsilon_0 > 0 \) is a constant depending on \( \sigma_k, \sigma_{k+1}, p_W \).

In words, we will say the estimator \( \hat{\mathbf{U}}_k \) achieves weak recovery of \( \mathbf{U}_k \) for \( \sigma_k > 1/\sqrt{\mathbf{I}_W} \). Obviously, the same threshold implies weak recovery of \( \mathbf{V}_k \). The arguments of [KXZ16, PWBM16] imply that no estimator (even using knowledge of \( p_W \)) can achieve weak recovery below the same threshold. We established that the information-theoretically optimal weak recovery threshold can be achieved adaptively with respect to the noise distribution \( p_W \).

### 5 Numerical illustration

In this section we conduct experiments to illustrate the performances of our algorithm.
5.1 Model setting

We generate observations $Y \in \mathbb{R}^{m \times n}$ according to the model (1.1), i.e. $Y = X + W$. For the signal matrix $X$, we let $X = (mn)^{1/4} U \Sigma V^\top$, where $\Sigma = \text{diag}(\sigma_1, \ldots, \sigma_r)$ is diagonal and $U \in \mathbb{R}^{m \times r}$ and $V \in \mathbb{R}^{n \times r}$ are independent uniformly random orthogonal matrices.

For the noise matrix $W$, we let $W = (W_{ij})_{i,j \leq m,j \leq n}$ with i.i.d. entries $W_{ij} \sim p_W$. We choose $p_W$ to be the a mixture of the Gaussian distributions $N(-\mu, 1)$ and $N(\mu, 1)$, whose density is

$$p_W(x) = \frac{1}{2} \left( \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2}} + \frac{1}{\sqrt{2\pi}} e^{-\frac{(x+\mu)^2}{2}} \right). \quad (5.1)$$

Parameter choice  For the data generating process, we choose $r \in \{1, 3\}$, $m = n \in \{200, 400, 800\}$, $\mu = 2$.

Under this choice of $\mu$,

$$\text{Var}(W_{11}) = 5, \quad I_W \approx 0.7256.$$  

For our algorithm, we let

$$h_n = 1.2 (mn)^{-1/5}, \quad h'_n = (mn)^{-1/7}, \quad \varepsilon = 0.001, \quad \delta = 0.01.$$  

In the case $r = 1$, we let

$$\sigma_1 \in \{0.2, 0.4, 0.6, \ldots, 4\}$$

In the case $r = 3$, we let

$$\sigma_1 \in \{0.2, 0.4, 0.6, \ldots, 4\}, \quad \sigma_2 = 0.8\sigma_1, \quad \sigma_3 = 0.6\sigma_1.$$  

For each choice of $r, n$ and $\Sigma$, we generate 50 instances of the matrices $X$ and $Y$, and apply our estimation algorithm to compute the empirical average of corresponding accuracy metrics.

5.2 Singular space recovery

We first consider the accuracy in recovering the left singular subspace (by symmetry, there is no loss of generality in considering the left, rather than the right singular subspace). We will compare our approach to classical principal component analysis, which does not denoise the matrix $Y$, and instead computes directly its singular value decomposition. Let the singular value decomposition of $Y$ be

$$Y = \bar{U} \Sigma^{(0)} (\bar{V})^\top.$$  

As usual, we denote by $\bar{U}_k \in \mathbb{R}^{m \times k}$ the sub-matrix containing the top $k$ left singular vectors of $Y$, and by $\bar{V}_k \in \mathbb{R}^{n \times k}$ the sub-matrix containing the top $k$ right singular vectors.

We compare the two approaches by computing $\sigma_{\min}(\bar{U}_i^\top U_i)$ and $\sigma_{\min}(\bar{U}_i^\top U_i)$ for $1 \leq i \leq r$. Note that Theorem 5 predicts the limit of $\sigma_{\min}(\bar{U}_i^\top U_i)$ to be given by $G(\sigma_i; I_W)$ (by taking $n \to \infty$, then $\varepsilon \to 0$, then $\delta \to 0$). Analogously, Theorem 8 predicts the limit of $\sigma_{\min}(\bar{U}_i^\top U_i)$ to be given by $G(\sigma_i; \text{Var}(W_{11})^{-1})$.

The results of this computation are reported in Figure 1 (for $r = 1$) and Figure 2 (for $r = 3$). The present approach outperforms substantially standard PCA, and its behavior is well captured by the asymptotic theory.
Figure 1. Singular space recovery in the case $r = 1$. The three plots refer to $m = n \in \{200, 400, 800\}$ respectively. Red circles are the average of $\sigma_{\text{min}}(\hat{U}_1^T U_1)$ over 50 realizations, and black curve is its asymptotic prediction, given by $G(\sigma_1; I_W)$. Blue ‘+’ are the average of $\sigma_{\text{min}}(\hat{U}_1^T U_1)$ over 50 realizations, and green curve is its asymptotic prediction, given by $G(\sigma_1; \text{Var}(W_{\text{11}})^{-1})$. 
Figure 2. Singular space recovery in the case $r = 3$. The three plots refer to $m = n \in \{200, 400, 800\}$ respectively. Red circles, triangles and squares are the averages of $\sigma_{\text{min}}(\hat{U}_i^T U_i)$ over 50 realizations for $i = 1, 2, 3$ respectively; black, purple and orange solid curves are their asymptotic predictions, given by $G(\sigma_i; I_W)$. Blue ‘+’, ‘x’ and ‘−’ are the average of $\sigma_{\text{min}}(\bar{U}_i^T U_i)$ over 50 realizations for $i = 1, 2, 3$ respectively; black, purple and orange dotted curves are their asymptotic predictions, given by $G(\sigma_i; \text{Var}(W_{11})^{-1})$. 

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5.3 Signal estimation

We next consider the problem of estimating the matrix $X$. As before, we compare our estimator $\hat{X}(Y)$ to a simpler one, denoted by $X(Y; \delta)$, which does not make use of the denoising step, and is defined below (this approach was originally proposed in [SN13]):

1. Construct the diagonal matrix $\Sigma$ with entries
   \[
   \Sigma_{i,i} = \begin{cases} 
   \sigma H^{-1}(\sigma^{-1} \Sigma_{i,i}) & \text{if } \Sigma_{i,i} \geq (1 + \delta)H(1), \\
   0 & \text{otherwise.} 
   \end{cases} 
   \] (5.2)

2. Return
   \[
   X(Y; \delta) = \left(\frac{mn}{4}\right)^{1/4}U \Sigma V^T. 
   \] (5.3)

The asymptotic theory predicts $(nm)^{-1/4}\|\hat{X}(Y) - X\|_{op}$ to converge to $\sigma_1 \wedge I_{1/2} - 1/2 W$, and $(nm)^{-1/4}\|X(Y) - X\|_{op}$ to converge to $\sigma_1 \wedge \text{Var}(W_{11})^{-1/2}$. Again, theory captures well the behavior of our experiments already at moderate sizes. Above the information-theoretic threshold $\sigma_1 = I_{1/2}$, the new approach outperforms standard PCA.

6 Proof of Theorem 3 (lower bound)

Throughout the proof, we let
\[
Q(\gamma; r, \eta) := \liminf_{M \to \infty} \liminf_{n \to \infty, m/n \to \gamma} \inf_{X \in \mathcal{F}_{m,n}(r, M, \eta)} \sup_{X} \left\{ \frac{1}{(mn)^{1/4}} \mathbb{E} \|\hat{X}(Y) - X\|_{op} \right\}. 
\] (6.1)

It suffices to show the following lower bound on the quantity $Q(\gamma; r, \eta)$:
\[
Q(\gamma; r, \eta) \geq \frac{1}{2} \sqrt{\frac{\gamma^{1/4}\gamma^{-1/4}}{I_W}}. 
\] (6.2)

By symmetry, we can assume $\gamma \geq 1$. Denote $v \in \mathbb{R}^n$ to be the vector such that $v_i = 1/\sqrt{n}$ for all $i \in [n]$. For any $c > 0$, define the set $S(c)$ as follows:
\[
S(c) := \{xv^T | x \in \mathbb{R}^m, \|x\|_{\infty} \leq c\}. 
\] (6.3)

Note that $S(c) \subseteq \mathcal{F}_{m,n}(r, c, \eta)$ for any constant $c > 0$. Let $\pi_{\Gamma,c}$ denote the one dimensional Gaussian distribution with mean 0 and variance $\Gamma > 0$ truncated on the interval $[-c, c]$. Now, draw the vector $x \in \mathbb{R}^m$ such that each coordinate of $x$ is independent following the distribution $\pi_{\Gamma,c}$. This induces a distribution on $S(c)$, for which we denote by $\mathbb{P}^{\Gamma,c}_{x}$.

To show Eq (6.2), we start by introducing the following Bayesian setting. Let $X$ be sampled from $\pi_{\Gamma,c}$ and $W$ be an independent matrix with i.i.d entries with distribution $p_W$. Set $Y = X + W$. With a slight abuse of notations, in the rest of the proof, we denote the joint distribution of $(X, Y)$ under this sampling scheme by $\mathbb{P}^{\Gamma,c}_{\pi}$. We use $\mathbb{E}^{\Gamma,c}_{\pi}$ to denote the expectation under $\mathbb{P}^{\Gamma,c}_{\pi}$. Now, for any $\gamma, r, M, \eta, n$, define the quantity
\[
R(\gamma; r, M, \eta, n) := \inf_{X} \sup_{X \in \mathcal{F}_{m,n}(r, M, \eta)} \left\{ \frac{1}{(mn)^{1/4}} \mathbb{E} \|\hat{X}(Y) - X\|_{op} \right\}. 
\] (6.4)
Figure 3. Matrix denoising under the operator norm loss in the case $r = 1$. The three plots refer to $m = n \in \{200, 400, 800\}$ respectively. Red circles are averages of $(nm)^{-1/4} \| \hat{X}(Y) - X \|_{\text{op}}$ over 50 realizations, and blue curve is its asymptotic prediction, given by $\sigma \wedge I_W^{-1/2}$. Blue ‘+’ are averages of $(nm)^{-1/4} \| X(Y; \delta) - X \|_{\text{op}}$ over 50 realizations, and green curve is its asymptotic prediction, given by $\sigma \wedge \text{Var}(W_{11})^{-1/2}$. 


Figure 4. Matrix denoising under the operator norm loss in the case $r = 3$. The three plots refer to $m = n \in \{200, 400, 800\}$ respectively. Red circles are the average of $(nm)^{-1/4}\|\hat{X}(Y) - X\|_{op}$ over 50 realizations, and blue curve is its asymptotic prediction, given by $\sigma \wedge I_{\hat{W}}^{-1/2}$. Blue ‘+’ are the average of $(nm)^{-1/4}\|\hat{X}(Y; \delta) - X\|_{op}$ over 50 realizations, and green curve is its asymptotic prediction, given by $\sigma \wedge \text{Var}(W_{11})^{-1/2}$. 
By the standard reduction argument of lower bounding minimax risk by Bayesian risk, we get that,

\[ R(\gamma; r, M, \eta, n) \geq \inf_X \left\{ \mathbb{E}^{\Gamma, c}_{\pi} \left[ \frac{1}{(mn)^{1/4}} \left\| \hat{X}(Y) - X \right\|_{\text{op}} \right] \right\} \]

\[ = \inf_X \left\{ \mathbb{E}^{\Gamma, c}_{\pi} \left[ \mathbb{E}^{\Gamma, c}_{\pi} \left[ \frac{1}{(mn)^{1/4}} \left\| \hat{X}(Y) - X \right\|_{\text{op}} | Y \right] \right] \right\} \]

\[ \geq \mathbb{E}^{\Gamma, c}_{\pi} \left[ \inf_X \left\{ \mathbb{E}^{\Gamma, c}_{\pi} \left[ \frac{1}{(mn)^{1/4}} \left\| \hat{X}(Y) - X \right\|_{\text{op}} | Y \right] \right\} \right] \quad \text{(6.5)} \]

Now that, \( \|v\|_2 = 1 \). Thus, when \( X \) takes the form \( X = xv^T \), any estimator \( \hat{X} \) satisfies,

\[ \left\| \hat{X}(Y) - X \right\|_{\text{op}} \geq \left\| (\hat{X}(Y) - X)v \right\|_2 = \left\| \hat{X}v - x \right\|_2. \quad \text{(6.6)} \]

Substituting the above estimate into Eq (6.5), we get the lower bound,

\[ R(\gamma; r, M, \eta, n) \geq \mathbb{E}^{\Gamma, c}_{\pi} \left[ \inf_{x \in \mathbb{R}^{m \times n}} \left\{ \mathbb{E}^{\Gamma, c}_{\pi} \left[ \frac{1}{(mn)^{1/4}} \left\| \hat{X}(Y)v - x \right\|_2 | Y \right] \right\} \right] \]

\[ \geq \mathbb{E}^{\Gamma, c}_{\pi} \left[ \inf_{x \in \mathbb{R}^{m}} \left\{ \mathbb{E}^{\Gamma, c}_{\pi} \left[ \frac{1}{(mn)^{1/4}} \left\| \hat{x}(Y) - x \right\|_2 | Y \right] \right\} \right] \quad \text{(6.7)} \]

To further lower bound the right-hand of Eq (6.7), we use the next lemma.

**Lemma 6.1.** Let \( Z = (Z_1, Z_2, \ldots, Z_n) \in \mathbb{R}^n \) be a random vector with independent coordinates. Then for any constant \( K > 0 \), any (non-random) vector \( a = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n \) and any \( t > 0 \), we have

\[ \mathbb{E} \|Z - a\|_2 \geq \left( 1 - K^2n^{-1}t^{-2} \right) + \left( \mathbb{E}L^2(Z, a; K) - nt \right)^{1/2}. \quad \text{(6.8)} \]

where the loss \( L(\cdot; \cdot; K) : \mathbb{R}^n \times \mathbb{R}^n \to R \) is defined by

\[ L(Z, a; K) = \left( \sum_{i=1}^n (Z_i - a_i)^2 \wedge K \right)^{1/2}. \quad \text{(6.9)} \]

**Proof** Denote by \( T_i = (Z_i - a_i)^2 \wedge K \) for \( i \in [m] \). Note first that, by definition, we have

\[ \|Z - a\|_2 \geq L(Z, a; K) = \left( \sum_{i=1}^n T_i \right)^{1/2} \geq \left( \frac{1}{n} \sum_{i=1}^n \mathbb{E}[T_i] - nt \right)^{1/2} \cdot \left( \sum_{i=1}^n T_i \geq \sum_{i=1}^n \mathbb{E}[T_i] - nt \right) \quad \text{(6.10)} \]

Using the definition of Eq (6.9) and taking expectation over both sides of Eq (6.10), we get

\[ \mathbb{E} \|Z - a\|_2 \geq \left( \mathbb{E}L^2(Z, a; K) - nt \right)^{1/2} + \mathbb{P} \left( \sum_{i=1}^n T_i \geq \sum_{i=1}^n \mathbb{E}[T_i] - nt \right) \quad \text{(6.11)} \]

By assumption, \( \{T_i\}_{i=1}^n \)’s are independent random variables with \( |T_i| \leq K \) almost surely. Thus, we have, \( \text{Var} \left( \sum_{i=1}^n T_i \right) \leq \sum_{i=1}^n \text{Var}(T_i) \leq nK^2 \). Now Markov’s inequality implies,

\[ \mathbb{P} \left( \sum_{i=1}^n T_i \geq \sum_{i=1}^n \mathbb{E}[T_i] - nt \right) \geq 1 - \mathbb{P} \left( \left| \frac{1}{n} \sum_{i=1}^n (T_i - \mathbb{E}[T_i]) \right| \geq t \right) \geq 1 - K^2n^{-1}t^{-2}. \quad \text{(6.12)} \]
Hence, substituting the above estimate into Eq (6.7), we get that for given $Y$, we can use Lemma 6.1 to show that, for any estimator $\hat{x}$, $\mathbb{E}_{\theta \in \mathbb{R}} \left[ \mathcal{L}(x_i | Y, \hat{x}) \right] = \mathcal{L}(x_i | Y)$ for all $i \in [m]$. Now, we turn to lower bound the RHS of Eq (6.7). For each $i \in [n]$, denote $Y_i = (Y_{i,1}, Y_{i,2}, \ldots, Y_{i,n})^T$ and $W_i = (W_{i,1}, W_{i,2}, \ldots, W_{i,n})^T$ to be the $i$th row of the matrix $Y$ and $W$ respectively. Then, since $Y = X + W$, this implies

$$Y_i = x_i v + W_i \quad \text{for all } i \in [m].$$

(6.13)

Thus, by the joint independence of $\{(x_i, W_i, Y_i)\}_{i=1}^m$, we know that the posterior distribution of $x$ given $Y$ has independent coordinates. Moreover, we have,

$$\mathcal{L}(x_i | Y) = \mathcal{L}(x_i | Y_i) \quad \text{for all } i \in [m].$$

(6.14)

Now, we can use Lemma 6.1 to show that, for any estimator $\hat{x} \in \mathbb{R}^m$ and $t, K > 0$:

$$\mathbb{E}_{\pi} \left[ \|\hat{x}(Y) - x\|_2 | Y \right] \geq (1 - K^2 m^{-1} t^{-2}) + \sum_{i=1}^m \mathbb{E}_{\pi} \left[ \|\hat{x}(Y) - x_i\|_2^2 \wedge K | Y_i \right] - mt + (6.15)

Hence, substituting the above estimate into Eq (6.7), we get that for $M \geq c$,

$$R(\gamma; r, M, \eta, n) \geq (1 - K^2 m^{-1} t^{-2}) + \mathbb{E}_{\pi} \left[ \inf_{\hat{x} \in \mathbb{R}^n} \left( \frac{1}{(mn)^{1/2}} \sum_{i=1}^m \mathbb{E}_{\pi} \left[ \|\hat{x} - x_i\|_2^2 \wedge K | Y_i \right] - \gamma^{1/2} t \right)^{1/2} \right].$$

(6.16)

Now, to further lower bound the RHS of Eq (6.16), we use [LCY12, Proposition 4, Section 6.4]. Note that a simple transformation of that result gives the result below (see e.g. [Duc18, Theorem 2]).

**Theorem 6.** Let $X = (X_1, X_2, \ldots, X_n)^T \in \mathbb{R}^n$ whose coordinates $X_i$ are independent random variables drawn from the same distribution $P$. Denote by $P_\theta$ the distribution of a shift of $P$ by $\theta$ for all $\theta \in \mathbb{R}$. Suppose that the families $\{P_{\theta}^{\otimes n}\}_{\theta \in \mathbb{R}}$ is a quadratic mean differentiable family, so that for some quantity $I$ and random variable $\Delta_n(X)$,

$$\log \frac{dP_{h/\sqrt{n}}^{\otimes n}(X)}{dP_{\otimes n}(X)} = h \Delta_n(X) - \frac{1}{2} h^2 I + op(1)$$

(6.17)

holds for all $h \in \mathbb{R}$. Now, denote $Y_i = \frac{1}{\sqrt{n}} h + X_i$ for $i \in [n]$ and $Y = (Y_1, Y_2, \ldots, Y_n)^T \in \mathbb{R}^n$. Fix $\Gamma > 0$. Denote $\pi^{\Gamma,c}$ to be the one dimensional Gaussian distribution with mean 0 and variance $\Gamma$, truncated to $[-c, c]$. Put $\pi^{\Gamma,c}$ as the prior distribution on $h$ and denote $\pi^{\Gamma,c}(\cdot | Y)$ to be the posterior of $h$ given $Y$. Then for any $\varepsilon > 0$, there exist $C = C(\varepsilon)$ and $N = N(\varepsilon)$ such that for $c \geq C$ and $n \geq N,$

$$\int \|G^{\Gamma}(\cdot | \mathbf{1}^{-1} Y) - \pi^{\Gamma,c}(\cdot | Y)\|_{TV} d\mathbb{P}_{n}(Y) \leq \varepsilon,$$

(6.18)

where in above we denote $G^{\Gamma}(\cdot | \mathbf{1}^{-1} Y)$ to be the Gaussian distribution

$$G^{\Gamma}(\cdot | \mathbf{1}^{-1} Y) = \mathcal{N}\left((I + \Gamma^{-1})^{-1} \Delta_n(Y), (I + \Gamma^{-1})^{-1}\right).$$

(6.19)

and $\mathbb{P}_n(Y)$ to be the marginal distribution of $Y$,

$$\bar{P}_n = \int P_{h/\sqrt{n}, n} d\pi^{\Gamma,c}(h).$$

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Now, fix $\Gamma > 0$ and $\varepsilon > 0$. By assumption, the coordinates of $Y = Y_i$ (cf. Eq. (6.13)) are i.i.d. with common distribution $P_\theta$ which is a shift of the density $p_W$ with finite Fisher information. Hence, Eq. (6.17) holds with $I = I_W$, and $\Delta_n(Y) = n^{-1/2}\sum_{i \leq n} p'_W(Y_i)/p_W(Y_i)$. Denote by $\pi^{\Gamma,c}(. \mid Y_i)$ the conditional distribution of $x_i$ given $Y_i$ and $G^\Gamma(\cdot \mid Y_i)$ to be the normal distribution as in Eq (6.19)

$$
G^\Gamma(\cdot \mid Y) = N((I_W + \Gamma^{-1})^{-1}\Delta_n(Y), (I_W + \Gamma^{-1})^{-1}).
$$

(6.20)

Recall our observation model is equivalent to the model in Eq (6.13). By Theorem 6, there exist some $c = c(\varepsilon) > 0$ and $n_0 = n_0(\varepsilon) \in \mathbb{N}$, such that the following holds for all $n \geq n_0(\varepsilon)$ and $i \in [m]$:

$$
\mathbb{E}_{\pi}^{\Gamma,c} \|G^\Gamma(\cdot \mid Y_i) - \pi^{\Gamma,c}(\cdot \mid Y_i)\|_{TV} \leq \varepsilon,
$$

(6.21)

Now, for each $i \in [m]$, define the event

$$
\Lambda_i := \{\|G^\Gamma(\cdot \mid Y_i) - \pi^{\Gamma,c}(\cdot \mid Y_i)\|_{TV} \leq \varepsilon^{1/2}\}.
$$

(6.22)

By Markov’s inequality, we know that

$$
\mathbb{P}_{\pi}^{\Gamma,c}(\Lambda_i) \geq 1 - \varepsilon^{-1/2}\mathbb{E}_{\pi}^{\Gamma,c} \|G^\Gamma(\cdot \mid Y_i) - \pi^{\Gamma,c}(\cdot \mid Y_i)\|_{TV} \geq 1 - \varepsilon^{1/2}.
$$

(6.23)

Now by definition of $\|\cdot\|_{TV}$, we know on event $\Lambda_i$, the following holds for all $x_i \in \mathbb{R}$

$$
\mathbb{E}_{\pi}^{\Gamma,c}[(\bar{x}_i - x_i)^2 \wedge K \mid Y_i] \geq \left(\mathbb{E}_{G}^{\Gamma,c}[(\bar{x}_i - x_i)^2 \wedge K \mid Y_i] - K\varepsilon^{1/2}\right)_{+}.
$$

(6.24)

Meanwhile, if we define the quantity

$$
J_{W,K}^{\Gamma} = \mathbb{E}_{T}[T^2 \wedge K] \text{ for } T \sim N(0, (I_W + \Gamma^{-1})^{-1}),
$$

then Anderson’s lemma [And55b] shows that, for all $i \in [m]$,

$$
\inf_{x \in \mathbb{R}} \mathbb{E}_{G}^{\Gamma}[(\bar{x}_i - x)^2 \wedge K \mid Y_i] \geq \mathbb{E}_{G}^{\Gamma}[(\bar{x}_i - \mathbb{E}_{G}^{\Gamma}[\bar{x}_i \mid Y_i])^2 \wedge K \mid Y_i] = J_{W,K}^{\Gamma},
$$

(6.25)

Thus Eq (6.24) and Eq (6.25) together imply that for all $i \in [m]$ and $x_i \in \mathbb{R}$,

$$
\mathbb{E}_{\pi}^{\Gamma,c}[(\bar{x}_i - x_i)^2 \wedge K \mid Y_i] \mathbb{1}\{\Lambda_i\} \geq \left(J_{W,K}^{\Gamma} - K\varepsilon^{1/2}\right)_{+} \mathbb{1}\{\Lambda_i\}.
$$

(6.26)

Thus, summing over $i \in [m]$ of Eq (6.26), we get that,

$$
\inf_{x \in \mathbb{R}^m} \sum_{i=1}^{m} \mathbb{E}_{\pi}^{\Gamma,c}[(\bar{x}_i - x_i)^2 \wedge K \mid Y_i] \geq \left(J_{W,K}^{\Gamma} - K\varepsilon^{1/2}\right)_{+} \sum_{i=1}^{m} \mathbb{1}\{\Lambda_i\}.
$$

(6.27)

Substituting the above bound into Eq (6.16), we get

$$
R(\gamma; r, M, \eta, n) \geq \gamma^{1/4} \cdot (1 - K^2m^{-1}t^{-2})_{+} \cdot \mathbb{E}_{\pi}^{\Gamma,c} \left[\left(J_{W,K}^{\Gamma} - K\varepsilon^{1/2}\right)_{+} \frac{1}{m} \sum_{i=1}^{m} \mathbb{1}\{\Lambda_i\} - t\right]^{1/2}.
$$

(6.28)

Now that $\Lambda_i \in \sigma(Y_i)$ by definition of $\{\Lambda_i\}_{i \in [m]}$, we see that the events $\{\Lambda_i\}_{i \in [m]}$ are mutually independent. Using Eq (6.23) and Hoeffding’s inequality, we immediately get for any $\varepsilon > 0$,

$$
\mathbb{P}_{\pi}^{\Gamma,c} \left(\frac{1}{m} \sum_{i=1}^{m} \mathbb{1}\{\Lambda_i\} \geq 1 - 2\varepsilon^{1/2}\right) \geq 1 - \exp(-4m\varepsilon).
$$

(6.29)
Thus, Markov’s inequality implies that,
\[
\mathbb{E}_n^{\Gamma,c} \left[ \left( (J_{W}^{\Gamma,K} - K\varepsilon^{1/2}) \cdot \frac{1}{m} \sum_{i=1}^{m} 1 \{ \Lambda_i \} - t \right) \right]^{1/2} \geq (1 - \exp(-4m\varepsilon)) \cdot \left( (J_{W}^{\Gamma,K} - K\varepsilon^{1/2})(1 - 2\varepsilon^{1/2})+ - t \right)^{1/2}.
\]
Substituting the above estimate into Eq (6.28), we get that,
\[
R(\gamma; r, M, \eta, n) \geq \gamma^{1/4} \cdot (1 - K^2m^{-1}t^{-2}) \cdot (1 - \exp(-4m\varepsilon)) \cdot \left( (J_{W}^{\Gamma,K} - K\varepsilon^{1/2})(1 - 2\varepsilon^{1/2})+ - t \right)^{1/2}.
\]
Now, we set \( t = m^{-1/3} \). Thus, we get for any \( \Gamma, K, \varepsilon > 0 \), there exist \( c = c(\varepsilon) \), \( n_0 = n_0(\varepsilon) > 0 \) such that for \( n \geq n_0 \) and \( M \geq c \),
\[
R(\gamma; r, M, \eta, n) \geq \gamma^{1/4}(1-K^2m^{-1/3})(1-\exp(-4m\varepsilon))^{1/2} \left( (J_{W}^{\Gamma,K} - K\varepsilon^{1/2})(1 - 2\varepsilon^{1/2})+ - m^{-1/3} \right)^{1/2}. \tag{6.30}
\]
Now, take \( n \to \infty \). We get for any \( \Gamma, K, \varepsilon > 0 \), there exists \( c = c(\varepsilon) > 0 \) such that for all \( M \geq c \),
\[
\liminf_{n \to \infty, m/n \to \gamma} R(\gamma; r, M, \eta, n) \geq \gamma^{1/4} \left( J_{W}^{\Gamma,K} - K\varepsilon^{1/2} \right)^{1/2} (1 - 2\varepsilon^{1/2})^{1/2}. \tag{6.31}
\]
Then, take \( M \to \infty \). This shows for any \( \Gamma, K, \varepsilon > 0 \),
\[
\lim_{M \to \infty} \liminf_{n \to \infty, m/n \to \gamma} R(\gamma; r, M, \eta, n) \geq \gamma^{1/4} \left( J_{W}^{\Gamma,K} - K\varepsilon^{1/2} \right)^{1/2}.
\tag{6.32}
\]
Lastly, we take \( \Gamma \to \infty, K \to \infty, K\varepsilon^{1/2} \to 0 \) on both sides of Eq (6.32). Since the LHS of the above is independent of \( \varepsilon, K, \Gamma > 0 \), and moreover we have,
\[
\lim_{\Gamma \to \infty, K \to \infty, K\varepsilon^{1/2} \to 0} \left( J_{W}^{\Gamma,K} - K\varepsilon^{1/2} \right)^{1/2} = 1_{W}^{-1/2}, \tag{6.33}
\]
and the theorem thus follows.

### 7 Proof outline of Theorem 4 (upper bound)

#### 7.1 Notation

For a family of random variables \( \{ X_{n,\alpha} \}_{n\in\mathbb{N}, \alpha \in A} \) and scalars \( \alpha_0, c \), we write
\[
\lim_{\alpha \to \alpha_0} \lim_{n \to \infty} X_{n,\alpha} = c \tag{7.1}
\]
if there exist a (non-random) function \( f : A \to \mathbb{R} \) and a neighborhood of \( \alpha_0 \), denoted by \( \mathcal{N}(\alpha_0) \) such that, for any fix \( \alpha \in \mathcal{N}(\alpha_0) \), we have, almost surely, \( \lim_{n \to \infty} X_{n,\alpha} = f(\alpha) \) and further \( \lim_{\alpha \to \alpha_0} f(\alpha) = c \).
We write
\[
\limsup_{\alpha \to \alpha_0} \limsup_{n \to \infty} X_{n,\alpha} \leq c \tag{7.2}
\]
if there exist a (non-random) function $f : A \to \mathbb{R}$ and a neighborhood $\mathcal{N}(\alpha_0)$ of $\alpha_0$, such that, for any fix $\alpha \in \mathcal{N}(\alpha_0)$, we have, almost surely, $\limsup_{n \to \infty} X_{n,\alpha} \leq f(\alpha)$, and further $\limsup_{\alpha \to \alpha_0} f(\alpha) = c$.

We write
\[
\lim_{\alpha \to \alpha_0} \limsup_{n \to \infty} X_{n,\alpha} = c \tag{7.3}
\]
if there exist two (non-random) function $f_1 : A \to \mathbb{R}$, $f_2 : A \to \mathbb{R}$ and a neighborhood $\mathcal{N}(\alpha_0)$ of $\alpha_0$, such that, for any fix $\alpha \in \mathcal{N}(\alpha_0)$, we have, almost surely, $f_1(\alpha) \leq \limsup_{n \to \infty} X_{n,\alpha} \leq f_2(\alpha)$ and $\lim_{\alpha \to \alpha_0} f_1(\alpha) = \lim_{\alpha \to \alpha_0} f_2(\alpha) = c$.

## 7.2 Proof outline and key lemmas

The proof consists of three steps. In the first step, we characterize the behavior of the denoised matrix $\hat{f}_Y(Y)$. Note that this theorem immediately implies the error estimate for $\hat{X}^{(*)}$ in the rank-unconstrained problem, namely Eq. (3.10).

**Theorem 7.** Let $\hat{f}_Y(\cdot), \hat{I}_{W,\varepsilon}$ be defined in Eq. (3.5). Then under the assumptions of Theorem 1, the following decomposition holds
\[
\hat{f}_Y(Y) = I_W X + \sqrt{I_W} Z + \Delta, \tag{7.4}
\]
where the matrices $Z \in \mathbb{R}^{m \times n}$ and $\Delta \in \mathbb{R}^{m \times n}$ satisfy the following properties:

1. The matrix $Z$ is a random matrix whose entries are i.i.d mean 0 and variance 1. Moreover, there exist constants $\varepsilon_0, C > 0$ independent of $m$ and $n$, such that almost surely for all $\varepsilon \leq \varepsilon_0$,
\[
\|Z\|_{\text{max}} \leq C\varepsilon^{-1}. \tag{7.5}
\]

2. The matrix $\Delta$ satisfies
\[
\lim_{\varepsilon \to 0^+} \limsup_{n \to \infty, m/n \to \gamma} \frac{1}{(mn)^{1/4}} \|\Delta\|_{\text{op}} = 0. \tag{7.6}
\]
Moreover, for some $\nu \in (0, 1]$, we have
\[
\lim_{\varepsilon \to 0^+} \lim_{n \to \infty, m/n \to \gamma} \frac{1}{(mn)^{(1+\nu)/4}} E \|\Delta\|_{\text{op}}^{1+\nu} = 0. \tag{7.7}
\]
Finally, $\hat{I}_{W,\varepsilon}$ satisfies
\[
\lim_{\varepsilon \to 0} \limsup_{n \to \infty, m/n \to \gamma} |\hat{I}_{W,\varepsilon} - I_W| = 0. \tag{7.8}
\]

An outline of the proof of this statement is presented in Section 7.3, with most technical work deferred to the appendices.

In the second step, we prove an “almost sure convergence version” of of Theorem 4. This is formulated in a more precise way in Lemma 7.1 below. Its proof is deferred to Appendix A.

**Lemma 7.1.** Let $(X_n)_{n \geq 1}$ be a sequence of matrices with $X_n \in F_{m,n}(r, M, \eta)$ where $m = m(n)$ is such that $\lim_{n \to \infty} m(n)/n = \gamma \in (0, \infty)$. Denote by $Y_n$ a noisy observation of $X_n$. Under the assumptions of Theorem 2, the estimator $\hat{X}(\cdot)$ satisfies
\[
\lim_{\delta \to 0} \lim \limsup_{n \to \infty} \frac{1}{(mn)^{1/4}} \|\hat{X}(Y_n) - X_n\|_{\text{op}} \leq (\gamma^{1/4} + \gamma^{-1/4}) I_W^{-1/2}. \tag{7.9}
\]
The proof of this lemma requires an analysis of the eigen-structure of the random matrix $\tilde{Y} = I_Y X + \sqrt{W} Z$ which is (by the previous theorem) a good approximation of $\tilde{f}_{Y,\varepsilon}(Y)$. Note that $\tilde{Y}$ is a low-rank deformation of a matrix with independent random entries, and a vast amount of work has been devoted to the study of such matrices over the last ten years, see e.g. [BAP05, BS06, Pau07, OMH13, BGN12, BKYY16, Din17]. Despite the wealth of information available in the literature, only the recent work of Xiucheng Ding [Din17] considers our setting, whereby the signal matrix $X$ is non-random. For the reader’s convenience provide an independent proof of the relevant random matrix theory estimates, which are stated in Section 7.4.

In the third step, we prove that the errors $\|\hat{X}(Y_n) - X_n\|_{op}/(mn)^{1/4}$ are uniformly integrable, thus allowing to translate the almost sure result to a result on the expected error.

Lemma 7.2. Within the same setting of Lemma 7.1, there exist some $c > 0, \nu \in (0,1]$ such that for any $\delta, \varepsilon \leq c$,

$$\limsup_{n \to \infty, m/n \to \gamma} \frac{1}{(mn)^{(1+\nu)/4}} \mathbb{E}\left[\|\hat{X}(Y_n) - X_n\|_{op}^{(1+\nu)}\right] < \infty. \quad (7.10)$$

The proof of the lemma is deferred to Appendix B.

Now, we are ready to prove Theorem 4. Note that, by Lemma 7.1 and Lemma 7.2, we know that, for any $\zeta > 0$, there exists some $\delta_0 > 0$ such that, for all $\delta < \delta_0$, there exists some $\varepsilon_0 = \varepsilon_0(\delta)$ such that for all $\varepsilon \leq \varepsilon_0$, we have almost surely,

$$\limsup_{n \to \infty} \frac{1}{(mn)^{1/4}} \|\hat{X}(Y_n; \varepsilon, \delta) - X_n\|_{op} \leq \max\{\gamma^{1/4}, \gamma^{-1/4}\} I_W^{-1/2} + \zeta. \quad (7.11)$$

(For the sake of clarity, we made explicit the dependence upon $\varepsilon, \delta$.) Moreover, for some $\nu > 0$,

$$\limsup_{n \to \infty} \frac{1}{(mn)^{(1+\nu)/4}} \mathbb{E}\left\{\|\hat{X}(Y_n; \varepsilon, \delta) - X_n\|_{op}^{(1+\nu)}\right\} < \infty. \quad (7.12)$$

Thus, by Eq (7.11) and Eq (7.12), we know that, for any $\zeta > 0$, there exists some $\delta_0 > 0$ such that, for all $\delta < \delta_0$, there exists some $\varepsilon_0 = \varepsilon_0(\delta)$ such that for all $\varepsilon \leq \varepsilon_0$,

$$\limsup_{n \to \infty} \frac{1}{(mn)^{1/4}} \mathbb{E}\|\hat{X}(Y_n; \varepsilon, \delta) - X_n\|_{op} \leq \max\{\gamma^{1/4}, \gamma^{-1/4}\} I_W^{-1/2} + \zeta. \quad (7.13)$$

Now notice that $X \mapsto \mathbb{E}\|\hat{X}(X + W; \varepsilon, \delta) - X\|_{op}$ is a continuous function of $X$ in the compact domain $F_{m,n}(r, M, \eta)$. Hence, there exists $X_n^*$ such that

$$\mathbb{E}\|\hat{X}(X_n^* + W; \varepsilon, \delta) - X_n^*\|_{op} = \sup_{X \in F_{m,n}(r, M, \eta)} \mathbb{E}\|\hat{X}(X + W; \varepsilon, \delta) - X\|_{op}. \quad (7.14)$$

Letting $Y_n^* = X_n^*$, we conclude that for any $\zeta > 0$, there exists some $\delta_0 > 0$ such that, for all $\delta < \delta_0$, there exists some $\varepsilon_0 = \varepsilon_0(\delta)$ such that for all $\varepsilon \leq \varepsilon_0$,

$$\limsup_{n \to \infty} \frac{1}{(mn)^{1/4}} \sup_{X \in F_{m,n}(r, M, \eta)} \mathbb{E}\|\hat{X}(Y; \varepsilon, \delta) - X\|_{op} = \limsup_{n \to \infty} \frac{1}{(mn)^{1/4}} \mathbb{E}\|\hat{X}(Y_n^*; \varepsilon, \delta) - X_n^*\|_{op} \leq \max\{\gamma^{1/4}, \gamma^{-1/4}\} I_W^{-1/2} + \zeta. \quad (7.15)$$

Hence, by taking $\varepsilon_n, \delta_n \downarrow 0$ sufficiently slowly, we obtain

$$\limsup_{n \to \infty} \frac{1}{(mn)^{1/4}} \sup_{X \in F_{m,n}(r, M, \eta)} \mathbb{E}\|\hat{X}(Y; \varepsilon_n, \delta_n) - X\|_{op} \leq \max\{\gamma^{1/4}, \gamma^{-1/4}\} I_W^{-1/2},$$

which coincides with the claim of the theorem.
7.3 Proof of Theorem 7

For any $\varepsilon > 0$, define the auxiliary function $f_{W,\varepsilon}(\cdot)$ by

$$f_{W,\varepsilon}(x) := -\frac{p_W'(x)}{p_W(x)} + \varepsilon. \quad (7.15)$$

Notice that $f_{W,\varepsilon}(x)$ is the ‘oracle’ denoiser, which uses knowledge of the noise distribution $p_W$. The proof of Theorem 7 proceed in two steps:

1. We analyze the matrix $f_{W,\varepsilon}(Y)$ obtained by applying the oracle denoiser, see Lemma 7.3 below.

2. We show that the denoiser $\hat{f}_Y$ behaves similarly to the oracle denoiser $f_{W,\varepsilon}$, for our purposes, cf. Lemma 7.4.

**Lemma 7.3.** Assume that the conditions of Theorem 4 hold. Then, we have the following decomposition of the matrix $f_{W,\varepsilon}(Y)$,

$$f_{W,\varepsilon}(Y) = I_W X + \sqrt{I_W} Z + \tilde{\Delta}, \quad (7.16)$$

where the matrices $Z \in \mathbb{R}^{m \times n}$ and $\tilde{\Delta} \in \mathbb{R}^{m \times n}$ satisfy the following properties:

1. The matrix $Z$ is a random matrix whose entries are i.i.d mean 0 and variance 1. Moreover, there exist constants $\varepsilon_0, C > 0$ independent of $m$ and $n$, such that almost surely for all $\varepsilon \leq \varepsilon_0$,

$$\|Z\|_{\text{max}} \leq C\varepsilon^{-1}. \quad (7.17)$$

2. The matrix $\tilde{\Delta}$ satisfies

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} \lim_{m/n \to \gamma} \frac{1}{(mn)^{1/4}} \|\tilde{\Delta}\|_{\text{op}} = 0. \quad (7.18)$$

Moreover, we have

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} \lim_{m/n \to \gamma} \frac{1}{(mn)^{1/2}} \mathbb{E} \|\tilde{\Delta}\|_{\text{op}}^2 = 0. \quad (7.19)$$

The basic idea in this lemma is to control various error terms in the Taylor expansion (1.3). A complete proof is deferred to Appendix C.

**Lemma 7.4.** Assume that the conditions of Theorem 4 hold. We have:

$$\lim_{\varepsilon \to 0^+} \lim_{n \to \infty} \frac{1}{(mn)^{1/4}} \|\hat{f}_Y(Y) - f_{W,\varepsilon}(Y)\|_{\text{op}} = 0, \quad (7.20)$$

and for some positive $\nu \in (0, 1]$,

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{(mn)^{(1+\nu)/4}} \mathbb{E} \|\hat{f}_Y(Y) - f_{W,\varepsilon}(Y)\|_{\text{op}}^{(1+\nu)} = 0. \quad (7.21)$$

Moreover, we have:

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{(mn)^{(1+\nu)/4}} \mathbb{E} \|\hat{f}_Y(Y) - f_{W,\varepsilon}(Y)\|_{\text{op}}^{(1+\nu)/4} = 0. \quad (7.22)$$

The key step in proving this lemma is to control $\max_{i \leq m, j \leq n} |\hat{f}_Y(Y_{ij}) - f_{W,\varepsilon}(Y_{ij})|$, which then allows to bound the operator norm difference $\|f_Y(Y) - f_{W,\varepsilon}(Y)\|_{\text{op}}$. In order to bound the maximum entry-wise difference, we use uniform bounds on the discrepancy between $\hat{p}_W$ and $p_W$, and $\hat{p}'_W$ and $p'_W$ on suitable intervals. We refer to Appendix D for the full proof.

It is clear that Theorem 7 follows form the last two lemmas.
7.4 A random matrix theory result

As mentioned above, a key step in proving Lemma 7.1 is to analyze the low-rank plus noise model that is derived in Theorem 7. In this section we state a random matrix theory result in order to prove Lemma 7.1, but which is potentially useful in other applications as well. Notice that, in Theorem 7, the signal and noise components are scaled—respectively—by $I_W$, and $\sqrt{I_W}$. However, these scalings can be absorbed into a scaling of the singular values of $X_n$, and an overall scaling of the matrix $Y_n$, which affects only the singular values on $Y_n$ and not its singular vectors. For the notational simplicity, we will remove these scalings in the present section.

We consider therefore a sequence of matrices $X_n \in \mathbb{R}^{m \times n}$ with rank $r$, and singular value decomposition:

$$X_n = (mn)^{1/4} \sum_{i=1}^r \sigma_i u_i v_i^T,$$

(7.23)

where $u_i = u_i(n) \in \mathbb{R}^m$ and $v_i = v_i(n) \in \mathbb{R}^n$ are singular vectors, and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r$ are singular values of $X_n/(mn)^{1/4}$ in decreasing order. This sequence of matrices is indexed by the number of columns, and we will assume the number of rows to be $m = m(n)$, such that $\lim_{n \to \infty} m(n)/n = \gamma$.

Let $W_n \in \mathbb{R}^{m \times n}$ be a random matrix with i.i.d entries $W_{i,j}$ satisfying $E W_n = 0$ and define $Y_n \in \mathbb{R}^{m \times n}$ to be

$$Y_n = X_n + W_n = (mn)^{1/4} \sum_{i=1}^r \sigma_i u_i v_i^T + W_n.$$

(7.24)

Introduce the singular value decomposition

$$Y_n = (mn)^{1/4} \sum_{i=1}^{m \wedge n} \hat{\sigma}_i \hat{u}_i \hat{v}_i^T,$$

(7.25)

where $\hat{u}_i = \hat{u}_i(n) \in \mathbb{R}^m$, $\hat{v}_i = \hat{v}_i(n) \in \mathbb{R}^n$ are left and right singular vectors and $\hat{\sigma}_1 \geq \hat{\sigma}_2 \geq \cdots$ the singular values of $Y_n/(mn)^{1/4}$ in decreasing order.

Recall the definition of $H(\sigma)$ in Eq. (3.3), and further define the functions

$$G^{(1)}(\sigma) = \left( \frac{1 - \sigma^{-4}}{1 + \gamma^{1/2} \sigma^{-2}} \right)^{1/2}, \quad \text{and} \quad G^{(2)}(\sigma) = \left( \frac{1 - \sigma^{-4}}{1 + \gamma^{-1/2} \sigma^{-2}} \right)^{1/2}.$$

(7.26)

The relevant random matrix theory estimates are stated below, and are essentially a restatement of a theorem in [Din17].

**Theorem 8.** Fix $r \in \mathbb{N}$ and singular values $\sigma_1, \ldots, \sigma_r$. Let $\{(u_1(n), \ldots, u_r(n))\}_{n \in \mathbb{N}}$ and $\{(v_1(n), \ldots, v_r(n))\}_{n \in \mathbb{N}}$ be two deterministic sequences of orthonormal sets of vectors, and define the matrices $X_n$ and $Y_n$ as in Eqs. (7.23), (7.24), with singular value decomposition (7.25). Suppose that the entries of $W_n$ are i.i.d with $E W_{i,j} = 0$ and $E W_{i,j}^2 = 1$ and moreover, $\sup_{n \geq 1} E\{|W_{ij}|^p\} < \infty$ for each $p \geq 1$. Assume $m = m(n)$ to be such that $\lim_{n \to \infty} m(n)/n = \gamma \in (0, \infty)$.

Then for each $i \in [r]$,

$$\hat{\sigma}_i \overset{a.s.}{\to} H(\sigma_i).$$

(7.27)

Moreover, assume that for some $k \in [r]$,

$$\sigma_1 > \sigma_2 > \cdots > \sigma_k > 1 \geq \sigma_{k+1} \geq \cdots \geq \sigma_r.$$
Then, we have for any $i \in [k]$,

$$
\langle \hat{u}_i, u_i \rangle \langle \hat{v}_i, v_i \rangle \xrightarrow{a.s.} G^{(1)}(\sigma_i)G^{(2)}(\sigma_i).
$$

(7.28)

Moreover, for any $i \in [k], j \in [r]$ we have

$$
\| \langle \hat{u}_i, u_j \rangle \| \xrightarrow{a.s.} \{ G^{(1)}(\sigma_i) \lor G^{(2)}(\sigma_i) \} 1 \{ i = j \}
$$

(7.29)

and

$$
\| \langle \hat{v}_i, v_j \rangle \| \xrightarrow{a.s.} \{ G^{(1)}(\sigma_i) \land G^{(2)}(\sigma_i) \} 1 \{ i = j \}.
$$

(7.30)

**Remark** Notice that Theorem 8 assumes the top $k$ singular values of the signal matrix $X_n/(mn)^{1/4}$ to be distinct. This is necessary in order to obtain convergence results of the form (7.28) that provide a one-to-one correspondence between singular vectors of the signal matrix, and singular vectors of the noisy matrix $Y_n$. However, no such assumption is made in our main result, Theorem 2, or in Lemma 7.1. Indeed, in the proof of Lemma 7.1 we remove the non-degeneracy assumption via a perturbation argument.

**Remark** Results analogous to Theorem 8 have been established under different settings [BGN12, BKYY16]. However, earlier work assumes that either $U_n = (u_1(n), \ldots, u_r(n))$ or $V_n = (v_1(n), \ldots, v_r(n))$ is random and independent from $W_n$. Such results cannot be used for the present application in which both $U_n$ and $V_n$ are fixed. The only case in which they can be applied is when $W_n$ has bi-unitarily invariant distribution which, for i.i.d. entries, implies that $W_n$ has Gaussian entries. Using invariance, we can effectively replace $U_n, V_n$ by uniform random rotations of the same matrices.

Ding [Din17] addressed the case of deterministic $U_n, V_n$, establishing the estimates in Theorem 8, by using earlier results of Knowles and Yin [KY17]. We propose here an independent proof both for readability and to make our assumptions more transparent.

Our proof of Theorem 8 (cf. Appendix E) follows a well-established strategy, see e.g. [BGN12]. We express the singular vectors and singular values of $Y_n$ in terms of the resolvent of the noise $W_n$, and the low-rank perturbation. The technical core of this type of argument is to control the resolvent, which we do using the moments’ method.

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**References**


A Proof of Lemma 7.1

A.1 Notation

We introduce the common notation used throughout the proof of Lemma 7.1. By Theorem 7, we know that, \( \hat{X}^{(0)}(Y) = \hat{f}_{Y;\varepsilon}(Y) \) has the decomposition below:

\[
\hat{X}^{(0)}(Y) = I_W X + \sqrt{I_W} Z + \Delta,
\]

(A.1)

where \( \Delta \) is a random matrix satisfying

\[
\lim_{\varepsilon \to 0} \limsup_{n \to \infty, m/n \to \gamma} \frac{1}{(mn)^{1/4}} \|\Delta\|_{op} = 0 \quad (A.2)
\]

and \( Z \) is some random matrix, whose entries are i.i.d bounded with mean 0 variance 1 and moreover, for some constants \( \varepsilon_0, C > 0 \) independent of \( m, n \), we have \( \|Z\|_{\max} \leq C\varepsilon^{-1} \) for \( \varepsilon \leq \varepsilon_0 \) and \( m, n \in \mathbb{N} \).

Denote the SVD decomposition of \( X \) to be

\[
X = (mn)^{1/4} U \Sigma V^T
\]

(A.3)

where \( U \in \mathbb{R}^{m \times r}, V \in \mathbb{R}^{n \times r} \) and \( \Sigma \in \mathbb{R}^{r \times r} \). Let \( k \) be the number of singular values of \( X \) strictly larger than the threshold \( I_W^{-1/2} \), i.e.,

\[
k = \left| \{ l : \Sigma_{l,l} > I_W^{-1/2} \} \right|,
\]

(A.4)

and \( \hat{k} \) be the rank of the estimator \( \hat{X}^{(0)} \), i.e.,

\[
\hat{k} = \left| \{ l : \hat{\Sigma}_{l,l} \neq 0 \} \right|.
\]

(A.5)

Recall the definition of \( \hat{U}, \hat{V}, \hat{\Sigma}, \hat{\Sigma}^{(0)} \) from Eq (3.6) and Eq (3.7). Define \( \hat{X}_k \in \mathbb{R}^{m \times n} \) by

\[
\hat{X}_k = (mn)^{1/4} \hat{U}_k \hat{\Sigma}_k \hat{V}_k^T.
\]

(A.6)

Define \( \tilde{X}^{(0)} \in \mathbb{R}^{m \times n} \) and its SVD decomposition as follows:

\[
\tilde{X}^{(0)} = I_W X + \sqrt{I_W} Z \quad \text{and} \quad \tilde{X}^{(0)} = (mn)^{1/4} \tilde{U} \tilde{\Sigma}^{(0)} \tilde{V}^T,
\]

(A.7)

Define \( \tilde{\Sigma}_k \in \mathbb{R}^{k \times k} \) to be a diagonal matrix with diagonal entries given by

\[
\tilde{\Sigma}_{i,i} = \begin{cases} \left( I_W^{-1/2} H^{-1}(I_W^{-1/2} \hat{\Sigma}_{i,i}^{(0)}) \right)^{1/2} & \text{if } \hat{\Sigma}_{i,i}^{(0)} \geq H(1) I_W^{-1/2} \\ 0 & \text{otherwise} \end{cases}
\]

(A.8)

Define the rank \( k \) matrix \( \widetilde{X}_k \in \mathbb{R}^{m \times n} \) by

\[
\widetilde{X}_k = (mn)^{1/4} \tilde{U}_k \tilde{\Sigma}_k \tilde{V}_k^T.
\]

(A.9)
A.2 Proof Outline

Our proof of Lemma 7.1 consists of three steps. In the first step, we argue that \( \hat{k} = k \) with high probability, and \( \bar{X}(Y) \approx \bar{X}_k(Y) \), as stated precisely in the next lemma. Its proof is given in Section A.2.1.

Lemma A.1. We have

\[
\lim_{\delta \to 0} \lim_{\epsilon \to 0} \lim_{n \to \infty, m/n \to \gamma} \frac{1}{(mn)^{1/4}} \| \bar{X}(Y) - \bar{X}_k(Y) \|_{op} = 0 \tag{A.10}
\]

In the second step, we show that \( \bar{X}_k(Y) \approx \tilde{X}_k(Y) \). This approximation is formulated rigorously in the next lemma, whose proof is given in Section A.2.2.

Lemma A.2. We have

\[
\lim_{\epsilon \to 0} \limsup_{n \to \infty, m/n \to \gamma} \frac{1}{(mn)^{1/4}} \| \bar{X}_k(Y) - \tilde{X}_k(Y) \|_{op} = 0 \tag{A.11}
\]

In the last step, we bound the operator norm distance between \( \tilde{X}_k(Y) \) and \( X \), as stated in the next lemma, whose proof is given in Section A.2.3.

Lemma A.3. We have

\[
\lim_{\epsilon \to 0} \limsup_{n \to \infty, m/n \to \gamma} \frac{1}{(mn)^{1/4}} \| \tilde{X}_k(Y) - X \|_{op} \leq I^{-1/2}_W (\gamma^{1/4} + \gamma^{-1/4}). \tag{A.12}
\]

Now the desired claim of Lemma 7.1 follows easily by Lemma A.1, Lemma A.2 and Lemma A.3.

A.2.1 Proof of Lemma A.1

We start by proving the following (recall the definition of \( \hat{k} \) at Eq (A.5))

\[
\lim_{\delta \to 0} \lim_{\epsilon \to 0} \lim_{n \to \infty, m/n \to \gamma} \hat{k} = k. \tag{A.13}
\]

To do so, we first analyze the limiting singular values of \( \bar{X}^{(0)}(Y) \). The lemma below is useful.

Lemma A.4. There exists some \( \epsilon_0 > 0 \) such that, for all \( \epsilon \leq \epsilon_0 \), we have,

1. For \( i \in [r] \), we have almost surely,

\[
\lim_{n \to \infty, m/n \to \gamma} \frac{1}{(mn)^{1/4}} \sigma_i(\bar{X}^{(0)}(Y)) = I^{1/2}_W H(I^{1/2}_W \sigma_i), \tag{A.14}
\]

2. For \( i = r + 1 \), we have almost surely,

\[
\limsup_{n \to \infty, m/n \to \gamma} \frac{1}{(mn)^{1/4}} \sigma_{r+1}(\bar{X}^{(0)}(Y)) \leq I^{1/2}_W H(1). \tag{A.15}
\]
Both Eq (A.14) and Eq (A.15) follow directly from Theorem 8.  

Note that \( \sigma_k > I_W^{-1/2} \) by definition of \( k \). Fix any \( \delta_0' \) such that,

\[
\sigma_k > (1 + \delta_0')I_W^{-1/2}. \tag{A.16}
\]

By Weyl’s inequality, we have the bound

\[
\max_{l \in \{k, k+1\}} \left| \sigma_l(\hat{X}^{(0)}(Y)) - \sigma_l(\tilde{X}^{(0)}(Y)) \right| \leq \|\Delta\|_{\text{op}}, \tag{A.17}
\]

Since by definition of \( \Delta \), we have (see Eq (A.2))

\[
\lim_{\varepsilon \to 0} \limsup_{n \to \infty, m/n \to \gamma} \frac{1}{(mn)^{1/4}} \|\Delta\|_{\text{op}} = 0, \tag{A.18}
\]

Lemma A.4, Eq (A.17) and Eq (A.18) imply that,

\[
\lim_{\varepsilon \to 0} \liminf_{n \to \infty, m/n \to \gamma} \frac{1}{(mn)^{1/4}} \sigma_k(\hat{X}^{(0)}(Y)) \geq I_W^{1/2}(1 + \delta_0') \tag{A.19}
\]

and

\[
\lim_{\varepsilon \to 0} \limsup_{n \to \infty, m/n \to \gamma} \frac{1}{(mn)^{1/4}} \sigma_{k+1}(\hat{X}^{(0)}(Y)) \leq I_W^{1/2}(1). \tag{A.20}
\]

Now Theorem 7 shows that

\[
\lim_{\varepsilon \to 0} \lim_{n \to \infty, m/n \to \gamma} \|\hat{I}_{W, \varepsilon} - I_W\| = 0. \tag{A.21}
\]

As the function \( \sigma \to H(\sigma) \) is strictly increasing on \([1, \infty)\), Eq (A.19), Eq (A.20) and Eq (A.21) together show that, for \( \delta > 0 \) sufficiently small satisfying \( (1 + \delta)H(1) < H(1 + \delta_0') \),

\[
\lim_{\varepsilon \to 0} \lim_{n \to \infty, m/n \to \gamma} \hat{k} = k. \tag{A.22}
\]

This proves the claim at Eq (A.13). In particular,

\[
\lim_{\delta \to 0} \lim_{\varepsilon \to 0} \lim_{n \to \infty, m/n \to \gamma} \frac{1}{(nm)^{1/4}} \|\hat{X}(Y) - \tilde{X}_k(Y)\|_{\text{op}} = 0, \tag{A.23}
\]

giving the desired claim of the Lemma.

### A.2.2 Proof of Lemma A.2

The main idea of our proof is the following: we view \( \hat{X}^{(0)}(Y) \) as a perturbation of \( \tilde{X}^{(0)}(Y) \), and then we establish a general matrix perturbation result to bound the difference \( \|\hat{X}(Y) - \tilde{X}(Y)\|_{\text{op}} \).

It turns out that such bound on \( \|\hat{X}(Y) - \tilde{X}(Y)\|_{\text{op}} \) gives the desired claim of the lemma.

We start by introducing our matrix perturbation result. Let \( A \in \mathbb{R}^{m \times n} \) be a matrix with singular value decomposition

\[
A = PSQ^T, \tag{A.24}
\]

where \( P \in \mathbb{R}^{m \times m} \), \( Q \in \mathbb{R}^{n \times n} \) and \( S \in \mathbb{R}^{m \times n} \). Define \( A_k \in \mathbb{R}^{m \times n} \) to be

\[
A_k = P_k S_k Q_k^T, \tag{A.25}
\]

\[\text{Proof.} \]
the best rank $k$ approximation for $A$ under any unitarily invariant norm \cite[Theorem 4.18]{SS90}. Now, for a function $f : \mathbb{R}_+ \to \mathbb{R}_+$, define the matrix $f(A_k) \in \mathbb{R}^{m \times n}$ by

$$f(A_k) = P_k f(S_k) Q_k^\top,$$

(A.26)

where $f(S_k) \in \mathbb{R}^{k \times k}$ is the diagonal matrix we get by applying $f$ entry-wisely to the diagonal of $S_k$.

Let $\widetilde{A} \in \mathbb{R}^{m \times n}$ denote a perturbation of matrix $A$:

$$\widetilde{A} = A + E,$$

(A.27)

where $E \in \mathbb{R}^{m \times n}$ is the error matrix. Define the matrix $\widetilde{A}_k$ and $f(\widetilde{A}_k)$ in an analogous way.

Intuitively, we expect that (under suitable assumptions on the matrix $A$ and function $f$) $f(\widetilde{A}_k) \approx f(A_k)$ when the error matrix $E$ is “small”. Theorem A.1 provides a bound of this type.

**Theorem A.1.** Let $\widetilde{A}, A, E \in \mathbb{R}^{m \times n}$ be matrices such that

$$\widetilde{A} = A + E,$$

(A.28)

Assume for some continuous function $f$ and some constants $L, \vartheta, \tau, \zeta > 0$ and $\alpha \in (0, 1]$, we have

1. The function $f$ is $(L, \alpha)$ Hölder continuous on $[\tau, \zeta]$, i.e., for all $x_1, x_2 \in [\tau, \zeta]$,

   $$|f(x_1) - f(x_2)| \leq L |x_1 - x_2|^{\alpha}. $$

   (A.29)

2. The following inequalities hold

   $$\zeta > \sigma_1(A), \sigma_k(A) > \max\{\sigma_{k+1}(A), \tau\} + \vartheta \quad \text{and} \quad \vartheta > 2 \|E\|_{\text{op}}. $$

   (A.30)

Then, the perturbation bound below holds for $f(A_k)$ and $f(\widetilde{A}_k)$:

$$\|f(A_k) - f(\widetilde{A}_k)\|_{\text{op}} \leq 4kL \|E\|_{\text{op}}^{\alpha} + \frac{2}{\vartheta} f(\sigma_k(A)) \|E\|_{\text{op}}.$$ 

(A.31)

We defer the proof of this theorem to Appendix G.

Define for $\varepsilon > 0$ the random variable $\tau_\varepsilon$ and the (random) function $f_\varepsilon$ by

$$\tau_\varepsilon = \hat{1}_{W, \varepsilon}^{1/2} H(1) \quad \text{and} \quad f_\varepsilon(\sigma) = \begin{cases} \hat{1}_{W, \varepsilon}^{1/2} H^{-1}(\hat{1}_{W, \varepsilon}^{1/2} \sigma) & \text{if } \sigma \geq \tau_\varepsilon \\ 0 & \text{otherwise} \end{cases}. $$

(A.32)

By definition, we have

$$\hat{X}(Y) = f_\varepsilon(\hat{X}^{(0)}(Y)) \quad \text{and} \quad \hat{X}(Y) = f_\varepsilon(\hat{X}^{(0)}(Y))$$

(A.33)

Now, viewing $\hat{X}^{(0)}(Y)$ as a perturbation of $\hat{X}^{(0)}(Y)$, we wish to use Theorem A.1 to give upper bounds on

$$\left\| \hat{X}(Y) - \hat{X}(Y) \right\|_{\text{op}} = \left\| f_\varepsilon(\hat{X}^{(0)}(Y)) - f_\varepsilon(\hat{X}^{(0)}(Y)) \right\|_{\text{op}}.$$ 

To apply Theorem A.1, we need to check two conditions. As our first step, we show that, for any $\varepsilon, \zeta > 0$, the function $f_\varepsilon$ is Hölder continuous on $[\tau_\varepsilon, \zeta]$. 30
Lemma A.5. For any $\varepsilon > 0, \zeta > 0$, $f_\varepsilon$ is $(4\hat{1}_{W_\varepsilon}^{1/2} \zeta^{3/4}, \frac{1}{4})$ Hölder continuous on $[\tau_\varepsilon, \zeta]$, i.e.,
\[
|f_\varepsilon(x_1) - f_\varepsilon(x_2)| \leq 4\hat{1}_{W_\varepsilon}^{1/2} \zeta^{3/4}|x_1 - x_2|^{1/4} \tag{A.34}
\]

Proof. For notational simplicity, we denote $h$ to be the constant
\[
h = \gamma^{1/2} + \gamma^{-1/2} \tag{A.35}
\]
By elementary computations, we have for all $x \geq H(1)$,
\[
H^{-1}(x) = \frac{1}{\sqrt{2}} \left( x^2 - h + ((x^2 - h)^2 - 4)^{1/2} \right)^{1/2}. \tag{A.36}
\]
Noting the elementary fact that $|x^{1/2} - y^{1/2}| \leq |x - y|^{1/2}$ for any $x, y \in \mathbb{R}_+$, we have for $x_1, x_2 \geq H(1)$
\[
|H^{-1}(x_1) - H^{-1}(x_2)| \leq \left( |x_1^2 - x_2^2| + \left| ((x_1^2 - h)^2 - 4)^{1/2} - ((x_2^2 - h)^2 - 4)^{1/2} \right| \right)^{1/2} \\
\leq |x_1^2 - x_2^2|^{1/2} + |(x_1^2 - h)^2 - (x_2^2 - h)^2|^{1/4} \leq 2|x_1 - x_2|^{1/4}|x_1 + x_2|^{3/4}.
\]
Hence, we have, for any $\varepsilon > 0$ and $x_1, x_2 \in [\tau_\varepsilon, \infty)$,
\[
|f_\varepsilon(x_1) - f_\varepsilon(x_2)| = \hat{1}_{W_\varepsilon}^{1/2} |H^{-1}(\hat{1}_{W_\varepsilon}^{1/2} x_1) - H^{-1}(\hat{1}_{W_\varepsilon}^{1/2} x_2)| \leq 2\hat{1}_{W_\varepsilon}^{1/2} |x_1 - x_2|^{1/4}|x_1 + x_2|^{3/4}. \tag{A.37}
\]
The estimate above gives the desired claim of Lemma A.5. \hfill \Box

Next, we bound the largest singular value and singular gap of $\widetilde{X}^{(0)}$.

Lemma A.6. There exist constants $\varepsilon_0, \vartheta_0, C_0 > 0$ independent of $m, n$ such that, for all $\varepsilon \leq \varepsilon_0$,
\[
\limsup_{n \to \infty, m/n \to \gamma} \frac{1}{(mn)^{1/4}} \sigma_1(\widetilde{X}^{(0)}) \leq C_0, \tag{A.38}
\]
and
\[
\liminf_{n \to \infty, m/n \to \gamma} \frac{1}{(mn)^{1/4}} \min\{\sigma_k(\widetilde{X}^{(0)}) - \sigma_{k+1}(\widetilde{X}^{(0)}) , \sigma_k(\widetilde{X}^{(0)}) - \tau_\varepsilon) \} \geq \vartheta_0. \tag{A.39}
\]

Proof. First of all, the first claim at Eq (A.38) follows directly from Lemma A.4. To prove the second claim at Eq (A.39), fix any $\delta'_0$ such that,
\[
\sigma_k > (1 + \delta'_0)\hat{I}_{W}^{-1/2}. \tag{A.40}
\]
Then Lemma A.4 implies that,
\[
\lim_{\varepsilon \to 0} \liminf_{n \to \infty, m/n \to \gamma} \frac{1}{(mn)^{1/4}} \sigma_k(\widetilde{X}^{(0)}) \geq \hat{I}_{W}^{1/2} H(1 + \delta'_0) \tag{A.41}
\]
and
\[
\lim_{\varepsilon \to 0} \limsup_{n \to \infty, m/n \to \gamma} \frac{1}{(mn)^{1/4}} \sigma_{k+1}(\widetilde{X}^{(0)}) \leq \hat{I}_{W}^{1/2} H(1). \tag{A.42}
\]
Now, by Theorem 7, we know the following convergence
\[
\lim_{\varepsilon \to 0} \limsup_{n \to \infty, m/n \to \gamma} |I_{W,\varepsilon} - I_W| = 0, \quad (A.43)
\]
this implies that,
\[
\lim_{\varepsilon \to 0} \limsup_{n \to \infty, m/n \to \gamma} |\tau_{\varepsilon} - I_W^{1/2}H(1)| = 0. \quad (A.44)
\]
Thus, if we set \(\vartheta_0\) to be
\[
\vartheta_0 = \frac{1}{2}I_W^{1/2} \left( H(1 + \delta_0') - H(1) \right) > 0, \quad (A.45)
\]
then Eq (A.41), Eq (A.42) and Eq (A.44) imply that
\[
\lim_{\varepsilon \to 0} \liminf_{n \to \infty, m/n \to \gamma} \left( \frac{1}{mn} \right)^{1/4} \min\{\sigma_k(\tilde{X}^{(0)}), \sigma_{k+1}(\tilde{X}^{(0)}), \sigma_k(\tilde{X}^{(0)}) - \tau_{\varepsilon}\} > \vartheta_0. \quad (A.46)
\]
This implies the second claim at Eq (A.39).

Now we are ready to apply Theorem A.1 to the matrices \(A = \tilde{X}^{(0)}(Y), \tilde{A} = \tilde{X}^{(0)}(Y)\) and the function \(f = f_{\varepsilon}\). Indeed pick \(\varepsilon_0, \vartheta_0, C_0\) such that the statement of Lemma A.6 holds. By Eq (A.2), we know for any \(\bar{\Delta} > 0\), there exists some \(\varepsilon_1 > 0\) such that for \(\varepsilon \leq \varepsilon_1\),
\[
\limsup_{n \to \infty, m/n \to \gamma} \frac{1}{(mn)^{1/4}} \|\bar{\Delta}\|_{op} \leq \bar{\Delta}. \quad (A.47)
\]
Note that by definition,
\[
\tilde{X}(Y) = f_{\varepsilon}(\tilde{X}^{(0)}(Y)) \quad \text{and} \quad \tilde{X}(Y) = f_{\varepsilon}(\tilde{X}^{(0)}(Y)). \quad (A.48)
\]
By Lemma A.5, Lemma A.6 and Theorem A.1, we know that, when \(\varepsilon \leq \varepsilon_0 \wedge \varepsilon_1\) and \(\bar{\Delta} < \vartheta_0/2\),
\[
\limsup_{n \to \infty, m/n \to \gamma} \frac{1}{(mn)^{1/4}} \|\bar{\Delta}\|_{op} \leq \frac{1}{2} \vartheta_0. \quad (A.49)
\]
Now, denote the quantity \(\tau\) and the function \(f\) to be
\[
\tau = I_W^{-1/2}H(1), \quad \tilde{f}(\sigma) = \begin{cases} I_W^{-1/2}H^{-1}(I_W^{-1/2}\sigma) & \text{if } \sigma \geq \tau \\ I_W^{-1/2} & \text{otherwise} \end{cases}. \quad (A.51)
\]
As Theorem 7 shows the following convergence
\[
\lim_{\varepsilon \to 0} \limsup_{n \to \infty, m/n \to \gamma} |I_{W,\varepsilon} - I_W| = 0, \quad (A.52)
\]
we get for any \(a > 0\),
\[
\lim_{\varepsilon \to 0} \limsup_{n \to \infty, m/n \to \gamma} (f_{\varepsilon}(a) - \tilde{f}(a)) \leq 0. \quad (A.53)
\]
Now, using Eq (A.52) and Eq (A.53), and taking $\varepsilon \to 0$ on both sides of Eq (A.50), we get that,
\[
\lim_{\varepsilon \to 0} \lim_{n \to \infty} \limsup_{m/n \to \gamma} \frac{1}{(mn)^{1/4}} \| \tilde{X}_k(Y) - \tilde{X}_k \|_{op} \leq 4kI_W^{-1/2}H^{-1/2}H^{-1} + \tilde{f}(C_0) \cdot \frac{2\Delta}{\vartheta_0}.
\]  
(A.54)

Note that the LHS of above is independent of $\Delta < \vartheta_0/2$. Taking $\Delta \to 0$ in Eq (A.54) gives us
\[
\lim_{\varepsilon \to 0} \lim_{n \to \infty} \limsup_{m/n \to \gamma} \frac{1}{(mn)^{1/4}} \| \tilde{X}_k(Y) - \tilde{X}_k \|_{op} = 0.
\]  
(A.55)

This proves the desired claim of Lemma A.2.

### A.2.3 Proof of Lemma A.3

In the proof below, we assume without loss of generality that $\gamma \leq 1$. Define the auxiliary matrix $\tilde{X}_k^{aux} \in \mathbb{R}^{m \times n}$ by
\[
\tilde{X}_k^{aux} = (mn)^{1/4} \tilde{U}_k \Sigma_k \tilde{V}_k^T.
\]  
(A.56)

We divide our proof of Lemma A.3 into two cases. In the first case, we consider the situation where the top singular values $\{\sigma_i\}_{i \in [k]}$ are distinct, i.e.,
\[
\sigma_1 > \sigma_2 > \ldots > \sigma_k > I_W^{-1/2}.
\]  
(A.57)

By triangle inequality, we know that,
\[
\frac{1}{(mn)^{1/4}} \| \tilde{X}_k - X \|_{op} \leq \frac{1}{(mn)^{1/4}} \| \tilde{X}_k - \tilde{X}_k^{aux} \|_{op} + \frac{1}{(mn)^{1/4}} \| \tilde{X}_k^{aux} - X \|_{op} = \| \tilde{\Sigma}_k - \Sigma_k \|_{op} + \| \tilde{U}_k \Sigma_k \tilde{V}_k^T - U \Sigma V^T \|_{op}.
\]  
(A.58)

Now, we upper bound the RHS of Eq (A.58) separately. The goal is to show that,
\[
\lim_{\varepsilon \to 0} \lim_{n \to \infty} \limsup_{m/n \to \gamma} \| \tilde{\Sigma}_k - \Sigma_k \|_{op} = 0.
\]  
(A.59)

and
\[
\lim_{\varepsilon \to 0} \lim_{n \to \infty} \limsup_{m/n \to \gamma} \| \tilde{U}_k \Sigma_k \tilde{V}_k^T - U \Sigma V^T \|_{op} \leq \max\{\gamma^{1/4}, \gamma^{-1/4}\} I_W^{-1/2}.
\]  
(A.60)

We start by proving Eq (A.59). By Lemma A.4 and Theorem 7, we know that for $i \in [k]$
\[
\lim_{n \to \infty} \lim_{m/n \to \gamma} \tilde{\Sigma}_{i,i}^{(0)} = \lim_{n \to \infty} \lim_{m/n \to \gamma} (mn)^{1/4} \sigma_i(\tilde{X}_k^{(0)}(Y)) = I_W^{1/2} H \left(I_W^{1/2} \sigma_i\right)
\]  
(A.61)

and
\[
\lim_{\varepsilon \to 0} \lim_{n \to \infty} \lim_{m/n \to \gamma} \hat{I}_{W,\varepsilon} = I_W.
\]  
(A.62)

Since $\sigma \to H^{-1}(\sigma)$ is continuous on $[H(1), \infty)$, Eq (A.61) and Eq (A.62) give for $i \in [k]$,
\[
\lim_{\varepsilon \to 0} \lim_{n \to \infty} \lim_{m/n \to \gamma} \hat{\Sigma}_{i,i} = I_W^{-1/2} H^{-1} \left(H \left(I_W^{1/2} \sigma_i\right)\right) = \sigma_i,
\]  
(A.63)
where the last identity uses the fact that \( \sigma_i > \Gamma_W^{1/2} \) for \( i \in [k] \). Hence, using Eq (A.63), we have,
\[
\lim_{\varepsilon \to 0} \lim_{n \to \infty, m/n \to \gamma} \| \bar{\Sigma}_k - \Sigma_k \|_{\text{op}} = \lim_{\varepsilon \to 0} \lim_{n \to \infty, m/n \to \gamma} \max_{i \in [k]} | \bar{\Sigma}_{i,i} - \sigma_i | = 0, \tag{A.64}
\]
which gives the desired bound in Eq (A.59).

Next, we show Eq (A.60). Fix \( \varepsilon > 0 \) first. Now we construct two auxiliary orthonormal matrices \( Q_u \in \mathbb{R}^{m \times m} \) and \( Q_v \in \mathbb{R}^{n \times n} \). First, define the two sets,
\[
S_u = \{ \tilde{u}_1, \ldots, \tilde{u}_k, u_1, \ldots, u_r \} \quad \text{and} \quad S_v = \{ \tilde{v}_1, \ldots, \tilde{v}_k, v_1, \ldots, v_r \}. \tag{A.65}
\]
Now, run the Gram-Schmidt orthogonalization process on the sets of vectors \( S_u \) and \( S_v \) separately. As the first \( k \) vectors of \( S_u \) (and same for \( S_v \)) are already orthogonal to each other, we may assume the output vectors of the process takes the form of
\[
S_u^{\text{GS}} = \{ \tilde{u}_1, \ldots, \tilde{u}_k, u_1', \ldots, u_r' \} \quad \text{and} \quad S_v^{\text{GS}} = \{ \tilde{v}_1, \ldots, \tilde{v}_k, v_1', \ldots, v_r' \}. \tag{A.66}
\]
The orthonormal matrices \( Q_u \) and \( Q_v \) are defined by any two orthonormal matrices whose first \( k + r \) columns are \( S_u^{\text{GS}} \) and \( S_v^{\text{GS}} \) respectively. Since both \( Q_u \in \mathbb{R}^{m \times m} \) and \( Q_v \in \mathbb{R}^{n \times n} \) are orthonormal matrices, we have,
\[
\left\| \bar{U}_k \Sigma_k \tilde{V}_k^T - U \Sigma V^T \right\|_{\text{op}} = \left\| Q_u^T \bar{U}_k \Sigma_k \tilde{V}_k^T Q_v - Q_u^T U \Sigma V^T Q_v \right\|_{\text{op}}. \tag{A.67}
\]
Denote the matrix \( \Gamma \in \mathbb{R}^{m \times n} \) to be
\[
\Gamma = Q_u^T \bar{U}_k \Sigma_k \tilde{V}_k^T Q_v - Q_u^T U \Sigma V^T Q_v. \tag{A.68}
\]
Then, by construction of \( Q_u \) and \( Q_v \), we know that the matrix \( \Gamma \) has 0 entries except its top left \((k + r) \times (k + r)\) sub-matrix. Denote this sub-matrix to be \( \Gamma_{\text{sub}} \in \mathbb{R}^{(k+r) \times (k+r)} \). For each \( i \in [k] \), define the angles \( \alpha_i, \beta_i \in (0, \pi/2) \) through the following:
\[
\cos \alpha_i = G^{(1)}(\sigma_i; \Gamma_W) \quad \text{and} \quad \cos \beta_i = G^{(2)}(\sigma_i; \Gamma_W), \tag{A.69}
\]
where we define the functions \( G^{(1)}(\sigma; t) \) and \( G^{(2)}(\sigma; t) \) for \( \sigma, t > 0, \sigma^2 t \geq 1 \) by
\[
G^{(1)}(\sigma; t) = \left( \frac{1 - t^2 \sigma^{-4}}{1 + \gamma^{1/2 t^{-1}} \sigma^{-2}} \right)^{1/2} \quad \text{and} \quad G^{(2)}(\sigma; t) = \left( \frac{1 - t \sigma^{-4}}{1 + \gamma^{1/2 t^{-1}} \sigma^{-2}} \right)^{1/2}. \tag{A.70}
\]
Moreover, introduce the following two by two matrices for \( i \in [k] \),
\[
\Delta^{(i)} = \sigma_i \begin{bmatrix} 1 - \cos \alpha_i \cos \beta_i & \sin \alpha_i \cos \beta_i \\ \sin \beta_i \cos \alpha_i & -\sin \alpha_i \sin \beta_i \end{bmatrix}. \tag{A.71}
\]
Now we prove that, almost surely we have

\[
\lim_{n \to \infty} \mathbf{R}^{\text{sub}} = \begin{bmatrix}
\Delta_{1,1}^{(1)} & 0 & \ldots & 0 & \Delta_{1,2}^{(1)} & 0 & \ldots & 0 & \ldots & 0 \\
0 & \Delta_{1,1}^{(2)} & \ldots & 0 & 0 & \Delta_{1,2}^{(2)} & \ldots & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & \ldots & 0 \\
0 & 0 & \ldots & \Delta_{1,1}^{(k)} & 0 & 0 & \ldots & \Delta_{1,2}^{(k)} & \ldots & 0 \\
0 & \Delta_{1,1}^{(1)} & 0 & \ldots & 0 & \Delta_{1,2}^{(1)} & \ldots & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & \ldots & 0 \\
0 & 0 & \ldots & \Delta_{1,1}^{(2)} & 0 & 0 & \ldots & \Delta_{1,2}^{(2)} & \ldots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & \ldots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & \ldots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & \ldots & 0 \\
\end{bmatrix}
\]  

(A.72)

Since \(\gamma \leq 1\), by Theorem 8, we can without loss of generality (by flipping the sign of \(\tilde{u}_i \in [k]\) and \(\tilde{v}_i \in [k]\) if necessary) assume that, for \(i \in [k]\) and \(j \in [r]\),

\[
\langle \tilde{u}_i, \tilde{u}_j \rangle \overset{\text{a.s.}}{\to} \begin{cases} 
\cos(\alpha_i) & \text{if } i \neq j \\
0 & \text{otherwise} 
\end{cases}
\quad \text{and} \quad
\langle \tilde{v}_i, \tilde{v}_j \rangle \overset{\text{a.s.}}{\to} \begin{cases} 
\cos(\beta_i) & \text{if } i \neq j \\
0 & \text{otherwise} 
\end{cases}
\]  

(A.73)

By definition of the Gram-Schmidt orthogonalization, we can without loss of generality (by flipping the sign of \(u'_i \in [k]\) and \(v'_i \in [k]\) if necessary) assume that, for \(i \in [k]\) and \(j \in [r]\),

\[
\langle u'_i, u'_j \rangle \overset{\text{a.s.}}{\to} \begin{cases} 
\sin(\alpha_i) & \text{if } i \neq j \\
0 & \text{otherwise} 
\end{cases}
\quad \text{and} \quad
\langle v'_i, v'_j \rangle \overset{\text{a.s.}}{\to} \begin{cases} 
\sin(\beta_i) & \text{if } i \neq j \\
0 & \text{otherwise} 
\end{cases}
\]  

(A.74)

Now, using Eq (A.73) and Eq (A.74), we see that,

\[
Q_u^\top \tilde{U}_k \overset{\text{a.s.}}{\to} \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{bmatrix}
\quad \text{and} \quad
Q_v^\top \tilde{V}_k \overset{\text{a.s.}}{\to} \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{bmatrix}
\]  

(A.75)
Now, Eq (A.72) follows by Eq (A.75), Eq (A.76) and the definition of $\Gamma_{\text{sub}}$ in Eq (A.68).

Following Eq (A.67) and Eq (A.72), this shows that,

$$ \lim_{n \to \infty, m/n \to \gamma} \| \tilde{U}_k \Sigma_k \tilde{V}_k^T - U \Sigma V^T \|_{\text{op}} = \lim_{n \to \infty, m/n \to \gamma} \| \Gamma_{\text{sub}} \|_{\text{op}} = \max_{i \in [k]} \| \Delta^{(i)} \|_{\text{op}}. $$

Now, we evaluate the RHS of Eq (A.77). To do so, for any $\alpha, \beta \in (0, \pi)$, define the matrix

$$ T(\alpha, \beta) = \begin{bmatrix}
1 - \cos \alpha & \cos \beta & \sin \alpha & \cos \beta \\
\sin \alpha & -\cos \beta & \cos \alpha & \sin \beta
\end{bmatrix}. $$

By elementary computations, the two singular values of $T(\alpha, \beta)$ are exactly

$$ 2 \cos (\alpha/2) \sin (\beta/2) \quad \text{and} \quad 2 \cos (\beta/2) \sin (\alpha/2), $$

and hence if we define the function $J(\alpha, \beta)$ by

$$ J(\alpha, \beta) = 2 \max \left\{ \cos \left( \frac{\alpha}{2} \right), \cos \left( \frac{\beta}{2} \right), \sin \left( \frac{\alpha}{2} \right) \right\}. $$

then we have for $\alpha, \beta \in (0, \pi)$, $\| T(\alpha, \beta) \|_{\text{op}} = J(\alpha, \beta)$. Hence, using Eq (A.77), we get that

$$ \lim_{n \to \infty, m/n \to \gamma} \| \tilde{U}_k \Sigma_k \tilde{V}_k^T - U \Sigma V^T \|_{\text{op}} = \max_{i \in [k]} \| \Delta^{(i)} \|_{\text{op}} = \max_{i \in [k]} 2\sigma_i J(\alpha_i, \beta_i) $$

Hence, to prove Eq (A.60), it suffices to show the estimate below,

$$ \max_{i \in [k]} 2\sigma_i J(\alpha_i, \beta_i) \leq \max \left\{ \gamma^{1/4}, \gamma^{-1/4} \right\} \zeta^{1/2}. $$

Denote the function $T : \mathbb{R}_+ \times [0, 1] \to \mathbb{R}_+$ to be

$$ T(t, y) := \left( 1 + \left( \frac{1 - y^2}{1 + ty} \right)^{1/2} \right) \left( 1 - \left( \frac{1 - y^2}{1 + ty} \right)^{1/2} \right). $$
By elementary computations, we know that, for \( i \in [k] \),
\[
2\sigma_i J(\alpha_i, \beta_i) = \sigma_i T_{1/2} \left( \max \{ \gamma^{1/2}, \gamma^{-1/2} \}, \sigma_i^{-2} I_W^{-1} \right).
\]  
(A.83)

Now we show that, for any \( t \geq 1 \) and \( y \in [0,1] \),
\[
T(t,y) \leq ty.
\]  
(A.84)

Indeed, introduce the function \( \hat{T}(t,y) := T(t,y) - ty \). Then, by computation, we have
\[
\frac{\partial}{\partial t} \hat{T}(t,y) = y \hat{T}(t,y),
\]  
(A.85)

where the function \( \hat{T}(t,y) \) is defined by
\[
\hat{T}(t,y) := \frac{1}{2}(1 - y^2)^{1/2} \left( (1 + t^{-1} y)^{-3/2} t^{-2} + (1 + ty)^{-3/2} \right)
+ \frac{1}{2} (1 - y^2) (1 + t^{-1} y)^{-3/2} (1 + ty)^{-3/2} (1 - t^{-2}) - 1.
\]  
(A.86)

Note that when \( t \geq 1 \), \( \hat{T}(t,y) \leq 0 \) as \( \hat{T}(1,1) = 0 \) and by inspection, \( \hat{T}(t,y) \) is nonincreasing in \( y \) for any fix \( t \in [1,\infty) \). Thus, by Eq (A.85), we know that, when \( t \geq 1 \),
\[
\frac{\partial}{\partial t} \hat{T}(t,y) \leq 0,
\]  
(A.87)

which implies that the function \( t \to \hat{T}(t,y) \) is decreasing in \( t \in [1,\infty) \) for any fix \( y \). Since \( \hat{T}(1,y) = T(1,y) - y = 0 \) for any \( y \in [0,1] \), this implies that \( \hat{T}(t,y) \leq 0 \) for any \( t \geq 1 \) and \( y \in [0,1] \), giving the desired claim at Eq (A.84). Now, with Eq (A.83) and Eq (A.84), we conclude that, for \( i \in [k] \),
\[
2\sigma_i J(\alpha_i, \beta_i) \leq \sigma_i \left( \max \{ \gamma^{1/2}, \gamma^{-1/2} \} \sigma_i^{-2} I_W^{-1} \right)^{1/2} = \max \{ \gamma^{1/4}, \gamma^{-1/4} \} I_W^{-1/2}.
\]  
(A.88)

This proves Eq (A.82), and as mentioned, it leads to the desired claim at Eq (A.60).

In the second case, we consider the situation where some elements of \( \{ \sigma_i \}_{i \in [k]} \) may coincide. We reduce ourselves to the first case using a perturbation argument outlined below. Indeed, for any \( \{ \epsilon_i \}_{i \in [k]} \) such that \( \{ \sigma_i + \epsilon_i \}_{i \in [k]} \) are distinct, we define
\[
X(\epsilon) = X + (mn)^{1/4} \sum_{i=1}^k \epsilon_i u_i v_i^T.
\]  
(A.89)

Now, for such \( \{ \epsilon_i \}_{i \in [k]} \), define \( \tilde{X}(\epsilon) \in \mathbb{R}^{m \times n} \) and its SVD decomposition of \( \tilde{X}(\epsilon) \) to be
\[
\tilde{X}(\epsilon) = I_W X(\epsilon) + \sqrt{I_W} Z \quad \text{and} \quad \tilde{X}(\epsilon) = (mn)^{1/4} \tilde{U}(\epsilon) \tilde{\Sigma}(\epsilon) \tilde{V}(\epsilon)^T,
\]  
(A.90)

Denote \( \tilde{\Sigma}_k(\epsilon) \in \mathbb{R}^{k \times k} \) to be the diagonal matrix with diagonal entries defined by,
\[
\tilde{\Sigma}_{i,i}(\epsilon) = \begin{cases} 
I_{W_{i,i}}^{-1/2} H^{-1}(I_{W_{i,i}}^{-1/2} \tilde{\Sigma}_{i,i}(\epsilon)) & \text{if } \tilde{\Sigma}_{i,i}(\epsilon) \geq H(1) I_{W_{i,i}}^{-1/2} \\
0 & \text{otherwise}
\end{cases}
\]  
(A.91)

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and the rank $k$ matrix $\widetilde{X}_k(t) \in \mathbb{R}^{m \times n}$ by
\[
\widetilde{X}_k(t) = (mn)^{1/4} \tilde{U}_k(t) \tilde{\Sigma}_k(t) \tilde{V}_k(t)^T.
\]
\text{(A.92)}

For notational shorthand, denote $i^\text{max} = \max_{i \in [k]} |t_i|$. Since $\sigma_k > \sigma_{k+1}$, by construction, when the value $i^\text{max}$ is sufficiently small, the set of the top $k$ singular values of $X(t)$ is precisely the set $\{\sigma_i + t_i \mid i \in [k]\}$. Thus by our choice of $\{t_i \mid i \in [k]\}$, the top $k$ singular values of $X(t)$ are pairwise different. Hence, we may use the established result in the first case to conclude that,
\[
\lim_{\varepsilon \to 0} \lim_{n \to \infty, m / n \to \gamma} \frac{1}{(mn)^{1/4}} \| \widetilde{X}_k(t) - X \|_{\text{op}} \leq \max \{ \gamma^{1/4}, \gamma^{-1/4} \} \Gamma W^{-1/2}.
\]
\text{(A.93)}

To prove our desired Eq (A.60), it suffices to show that,
\[
\lim_{\varepsilon \to 0} \lim_{\omega \to 0} \lim_{n \to \infty, m / n \to \gamma} \frac{1}{(mn)^{1/4}} \| \widetilde{X}_k(t) - \widetilde{X}_k \|_{\text{op}} = 0.
\]
\text{(A.94)}

To prove Eq (A.94), we use the same idea as that of proving Lemma A.2. Viewing $\widetilde{X}^{(0)}(t)$ as a perturbed version of $\widetilde{X}^{(0)}$, we may use the perturbation result Theorem A.1 to upper bound
\[
\| \widetilde{X}(t) - \widetilde{X} \|_{\text{op}} = \| f_\varepsilon \left( \widetilde{X}^{(0)}(t) \right) - \widetilde{f}_\varepsilon \left( \widetilde{X}^{(0)} \right) \|_{\text{op}},
\]
where we recall the definition of $f_\varepsilon$ at Eq (A.32). It turns out that such bound suffices for our proof of Eq (A.94).

To be more concrete, note first that, by definition
\[
\| \widetilde{X}(t) - \widetilde{X} \|_{\text{op}} = \varepsilon^\text{max}.
\]
\text{(A.96)}

Pick $\varepsilon_0, \vartheta_0, C_0 > 0$ such that the statement of Lemma A.6 holds. Recall $f_\varepsilon$ at Eq (A.32). Since
\[
\widetilde{X}(t) = f_\varepsilon \left( \widetilde{X}^{(0)}(t) \right) \quad \text{and} \quad \widetilde{X} = f_\varepsilon \left( \widetilde{X}^{(0)} \right).
\]
\text{(A.97)}

Lemma A.5, Lemma A.6 and Theorem A.1 imply that when $\varepsilon \leq \varepsilon_0$ and $\varepsilon^\text{max} < \vartheta_0/2$,
\[
\limsup_{n \to \infty, m / n \to \gamma} \frac{1}{(mn)^{1/4}} \| \widetilde{X}_k(t) - \widetilde{X}_k \|_{\text{op}} \leq \left( \limsup_{n \to \infty, m / n \to \gamma} \tilde{I}_W^{1/4} \right) \cdot 4kC_0^{3/4}(\varepsilon^\text{max})^{1/4} + \left( \limsup_{n \to \infty, m / n \to \gamma} f_\varepsilon(C_0) \right) \cdot \frac{2\varepsilon^\text{max}}{\vartheta_0}
\]
\text{(A.98)}

Recall $\tilde{f}$ at Eq (A.51). By Eq (A.53), if we take limit $\varepsilon \to 0$ in Eq (A.98), we get
\[
\lim_{\varepsilon \to 0} \limsup_{n \to \infty, m / n \to \gamma} \frac{1}{(mn)^{1/4}} \| \widetilde{X}_k(t) - \widetilde{X}_k \|_{\text{op}} \leq 4k\tilde{I}_W^{1/4} C_0^{3/4} (\varepsilon^\text{max})^{1/4} + \tilde{f}(C_0) \cdot \frac{2\varepsilon^\text{max}}{\vartheta_0}
\]
\text{(A.99)}

Take $t \to 0$ on both sides of Eq (A.99). This shows
\[
\lim_{\varepsilon \to 0, \omega \to 0} \limsup_{n \to \infty, m / n \to \gamma} \frac{1}{(mn)^{1/4}} \| \widetilde{X}_k(t) - \widetilde{X}_k \|_{\text{op}} = 0
\]
\text{(A.100)}

the desired claim at Eq (A.94). As mentioned, this proves Eq (A.60).
B  Proof of Lemma 7.2

First, note that, by Hölder’s inequality, we have for any \( \nu \in (0, 1] \):
\[
    \frac{1}{(mn)^{(1+\nu)/4}} \mathbb{E} \left[ \| \widehat{X}(\nu) - X \|_{op}^{(1+\nu)} \right] \leq \frac{2}{(mn)^{(1+\nu)/4}} \left( \mathbb{E} \left[ \| \widehat{X}(\nu) \|_{op}^{(1+\nu)} \right] + \mathbb{E} \left[ \| X \|_{op}^{(1+\nu)} \right] \right). \tag{B.1}
\]
Since \( X \in F_{m,n}(r, M, \eta) \), we know that, for any \( \nu \in (0, 1] \),
\[
    \frac{1}{(mn)^{(1+\nu)/4}} \mathbb{E} \left[ \| X \|_{op}^{(1+\nu)} \right] \leq M^{(1+\nu)} < \infty. \tag{B.2}
\]
Hence, using Eq (B.1), it suffices to show for some \( \nu \in (0, 1] \), we have for any \( \varepsilon, \delta \) small enough
\[
    \limsup_{n \to \infty, m/n \to \gamma} \frac{1}{(mn)^{(1+\nu)/4}} \mathbb{E} \left[ \| \widehat{X}(\nu) \|_{op}^{(1+\nu)} \right] < \infty. \tag{B.3}
\]
To show Eq (B.3), we start by providing a deterministic upper bound on \( \| \widehat{X}(\nu) \|_{op} \), which is,
\[
    \| \widehat{X}(\nu) \|_{op} \leq \varepsilon^{-1} \| \widehat{X}(0) \|_{op}. \tag{B.4}
\]
Indeed, we first note that,
\[
    \| \widehat{X}(\nu) \|_{op} = \begin{cases} \hat{i}_{W,\varepsilon}^{-1/2} H^{-1} \left( \hat{i}_{W,\varepsilon}^{-1/2} \| \widehat{X}(0) \|_{op} \right) & \text{if } \| \widehat{X}(0) \|_{op} \geq (1 + \delta) H(1) \hat{i}_{W,\varepsilon}^{1/2} \text{ otherwise.} \\
0 & \text{if } \| \widehat{X}(0) \|_{op} \geq (1 + \delta) H(1) \hat{i}_{W,\varepsilon}^{1/2} \end{cases} \tag{B.5}
\]
Next, by inspection \( H(\sigma) \geq \sigma \) for \( \sigma \geq 1 \). Thus \( H^{-1}(\sigma) \leq \sigma \) for \( \sigma > H(1) \). Hence, Eq (B.5) implies
\[
    \| \widehat{X}(\nu) \|_{op} \leq \hat{i}_{W,\varepsilon}^{-1} \| \widehat{X}(0) \|_{op} \tag{B.6}
\]
Since by definition \( \hat{i}_{W,\varepsilon} \geq \varepsilon \), this implies the desired Eq (B.4). Now, Eq (B.4) implies for \( \nu \in (0, 1] \),
\[
    \| \widehat{X}(\nu) \|_{op}^{(1+\nu)} \leq \varepsilon^{-1} \| \widehat{X}(0) \|_{op}^{(1+\nu)}. \tag{B.7}
\]
Now to show Eq (B.3), it suffices to show for some \( \nu \in (0, 1], c > 0 \), we have for any \( \varepsilon, \delta \leq c \)
\[
    \limsup_{n \to \infty, m/n \to \gamma} \frac{1}{(mn)^{(1+\nu)/4}} \mathbb{E} \left[ \| \widehat{X}(0) \|_{op}^{(1+\nu)} \right] < \infty. \tag{B.8}
\]
Now, we prove Eq (B.8). By Theorem 7, there exist some constants \( \varepsilon_0 > 0, \nu_0 \in (0, 1] \) such that for any \( \varepsilon \leq \varepsilon_0 \), the matrix \( \widehat{X}(0) = \hat{f}(\nu) \) has the decomposition:
\[
    \widehat{X}(0) = I_W X + \sqrt{I_W} Z + \Delta, \tag{B.9}
\]
where \( Z \) satisfies Eq (7.5) and \( \Delta \) satisfies Eq (7.7) for \( \nu = \nu_0 \). Fix this \( \varepsilon_0, \nu_0 \). By Hölder’s inequality,
\[
    \frac{1}{(mn)^{(1+\nu_0)/4}} \mathbb{E} \left[ \| \widehat{X}(0) \|_{op}^{(1+\nu_0)} \right] \leq \frac{3}{(mn)^{(1+\nu_0)/4}} \left( \| I_W X \|^2_{op} + \| I_W \|^2/2 \mathbb{E} \| Z \|^2_{op} + \mathbb{E} \| \Delta \|^2_{op} \right). \tag{B.10}
\]
First, since $X \in F_{m,n}(r,M,\eta)$, we know that
\[
\limsup_{n \to \infty, m/n \to \gamma} \frac{1}{(mn)^{(1+\nu_0)/4}} \|X\|_{\text{op}}^{(1+\nu_0)} \leq M^{(1+\nu_0)}. \tag{B.11}
\]
Next, when $\varepsilon \leq \varepsilon_0$, we know from Eq (7.5) that $\|Z\|_{\text{max}} < C\varepsilon^{-1}$ for some constant $C > 0$ independent of $m, n, \varepsilon$. Thus, when $\varepsilon \leq \varepsilon_0$, \[AGZ09\]
\[
\limsup_{n \to \infty, m/n \to \gamma} \frac{1}{(mn)^{(1+\nu_0)/4}} \mathbb{E} \|Z\|_{\text{op}}^{(1+\nu_0)} = (\gamma^{1/4} + \gamma^{-1/4})^{(1+\nu_0)}. \tag{B.12}
\]
Finally according to Eq (7.7), we know for sufficiently small $\varepsilon$,
\[
\limsup_{n \to \infty, m/n \to \gamma} \frac{1}{(mn)^{(1+\nu_0)/4}} \mathbb{E} \|\Delta\|_{\text{op}}^{(1+\nu_0)} < \infty. \tag{B.13}
\]
Substituting Eq (B.11), Eq (B.12) and Eq (B.13) into Eq (B.10), one proves Eq (B.8). As mentioned, Eq (B.8) implies Eq (B.3), which implies the desired Lemma 7.2.

C Proof of Lemma 7.3

To start with, since $Y_{i,j} = X_{i,j} + W_{i,j}$ for $i \in [m], j \in [n]$, Taylor’s expansion gives
\[
f_{W,\varepsilon}(Y_{i,j}) = f_{W,\varepsilon}(W_{i,j}) + X_{i,j} f'_{W,\varepsilon}(W_{i,j}) + \frac{1}{2} X_{i,j}^2 f''_{W,\varepsilon}(W_{i,j} + \theta_{i,j}), \tag{C.1}
\]
for some $\theta_{i,j}$ satisfying $|\theta_{i,j}| \leq |X_{i,j}|$. Now, denote $\bar{\mu}_\varepsilon, \bar{\nu}_\varepsilon$ and $\bar{\nu}_\varepsilon^2$ to be
\[
\bar{\mu}_\varepsilon = \mathbb{E}[f_{W,\varepsilon}(W_{i,j})], \quad \bar{\nu}_\varepsilon = \mathbb{E}[f_{W,\varepsilon}'(W_{i,j})] \quad \text{and} \quad \bar{\nu}_\varepsilon^2 = \text{Var}(f_{W,\varepsilon}(W_{i,j})). \tag{C.2}
\]
A key observation is that $\bar{\mu}_\varepsilon = 0$. Thus, if we define
\[
Z := \bar{\nu}_\varepsilon^{-1} (f_{W,\varepsilon}(W) - \bar{\mu}_\varepsilon) = \bar{\nu}_\varepsilon^{-1} f_{W,\varepsilon}(W) , \tag{C.3}
\]
Taylor’s expansion at Eq (C.1) implies the following decomposition,
\[
f_{W,\varepsilon}(Y) = I_W X + \sqrt{I_W} Z + \bar{\Delta}^{(1)} + \bar{\Delta}^{(2)} + \bar{\Delta}^{(3)} + \bar{\Delta}^{(4)}, \tag{C.4}
\]
where we introduce the matrices $\bar{\Delta}^{(1)}, \bar{\Delta}^{(2)}, \bar{\Delta}^{(3)}, \bar{\Delta}^{(4)} \in \mathbb{R}^{m \times n}$ to be:
\[
\bar{\Delta}^{(1)}_{i,j} := (f_{W,\varepsilon}'(W_{i,j}) - \bar{\nu}_\varepsilon) X_{i,j}, \quad \bar{\Delta}^{(2)}_{i,j} := \frac{1}{2} X_{i,j}^2 f''_{W,\varepsilon}(W_{i,j} + \theta_{i,j})
\]
and
\[
\bar{\Delta}^{(3)}_{i,j} := (\bar{\nu}_\varepsilon - \sqrt{I_W}) Z_{i,j}, \quad \bar{\Delta}^{(4)}_{i,j} := (\bar{\nu}_\varepsilon^2 - I_W) X_{i,j}.
\]
Define the matrix $\bar{\Delta}$ by
\[
\bar{\Delta} := \bar{\Delta}^{(1)} + \bar{\Delta}^{(2)} + \bar{\Delta}^{(3)} + \bar{\Delta}^{(4)}. \tag{C.5}
\]
In the rest of the proof, we show that $Z$ and $\bar{\Delta}$ satisfy the statements of the lemma.
Consider the matrix $Z$ first. By definition, $Z$ is a random matrix whose entries are i.i.d mean $0$ and variance $1$. Moreover, by dominated convergence theorem

$$\lim_{\varepsilon \to 0} I_{W,\varepsilon} = I_W \quad \text{and} \quad \lim_{\varepsilon \to 0} \bar{p}_{\varepsilon} = I_W. \quad (C.6)$$

Thus for some $\varepsilon_0 > 0$, we have $(\bar{p}_{\varepsilon})^{-1} \leq 2I_W^{-1/2}$ for all $\varepsilon \leq \varepsilon_0$. Hence, for $\varepsilon \leq \varepsilon_0$,

$$\|Z\|_{\text{max}} \leq \bar{p}_{\varepsilon}^{-1} \|f_{W,\varepsilon}(\cdot)\|_{\infty} \leq 2\varepsilon^{-1/2}I_W^{-1/2} \|p_{W}(\cdot)\|_{\infty}. \quad (C.7)$$

Since $\|p_{W}(\cdot)\|_{\infty} \leq M_2$ by Assumption A2, this shows that for some constant $C > 0$

$$\|Z\|_{\text{max}} \leq C\varepsilon^{-1} \quad (C.8)$$

for $\varepsilon \leq \varepsilon_0$. Together, we prove that $Z$ satisfies the statements of the lemma.

Next, we consider the matrix $\Delta$. We need to prove that $\Delta$ satisfies Eq (7.18) and Eq (7.19). To start with, by triangle inequality, we have,

$$\|\bar{\Delta}\|_{\text{op}} \leq \|\bar{\Delta}^{(1)}\|_{\text{op}} + \|\bar{\Delta}^{(2)}\|_{\text{op}} + \|\bar{\Delta}^{(3)}\|_{\text{op}} + \|\bar{\Delta}^{(4)}\|_{\text{op}}. \quad (C.9)$$

and

$$\|\bar{\Delta}\|^2 \leq 4 \left[ \|\bar{\Delta}^{(1)}\|^2_{\text{op}} + \|\bar{\Delta}^{(2)}\|^2_{\text{op}} + \|\bar{\Delta}^{(3)}\|^2_{\text{op}} + \|\bar{\Delta}^{(4)}\|^2_{\text{op}} \right]. \quad (C.10)$$

Thus, it suffices to show that for any $j \in [4],$

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty, m/n \to \gamma} \frac{1}{(mn)^{1/4}} \|\bar{\Delta}^{(j)}\|_{\text{op}} = 0 \quad (C.11)$$

and for any $j \in [4],$

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty, m/n \to \gamma} \frac{1}{(mn)^{1/2}} \|\bar{\Delta}^{(j)}\|^2_{\text{op}} = 0. \quad (C.12)$$

We show the proof of Eq (C.11) and Eq (C.12) for $\bar{\Delta}^{(1)}$, $\bar{\Delta}^{(2)}$, $\bar{\Delta}^{(3)}$ and $\bar{\Delta}^{(4)}$ one by one in the four consecutive paragraphs below.

**Proof of Eq (C.11) and Eq (C.12) for $\bar{\Delta}^{(1)}$.** The key part of the proof is to show the following moment bounds on $\|\bar{\Delta}^{(1)}\|_{\text{op}}$: there exists some $\varepsilon_0 > 0$ such that for any $k \geq 2$, there exists a constant $C_k > 0$ independent of $m, n$ such that, for all $m, n$ and $\varepsilon \leq \varepsilon_0$,

$$E\|\bar{\Delta}^{(1)}\|^k_{\text{op}} \leq C_k(m \vee n)^{(1/2-n)k} \log(m \vee n)^k \varepsilon^{-2k}. \quad (C.13)$$

Deferring the proof of Eq (C.13) at the sequel of the paragraph, we first show how Eq (C.13) immediately implies that Eq (C.11) and Eq (C.12) hold for $j = 1$. First, pick some $k \geq 2$ such that $\eta k \geq 2$. Markov’s inequality and Eq (C.13) imply for some constant $C_k > 0$, the inequality below holds for any $\delta > 0$,

$$P \left( \frac{1}{(mn)^{1/4}} \|\bar{\Delta}^{(1)}\|_{\text{op}} \geq \delta \right) \leq \delta^{-k}E \left( \frac{1}{(mn)^{k/4}} \|\bar{\Delta}^{(1)}\|^k_{\text{op}} \right) \leq C_k(m \vee n)^{-\eta k} \log(m \vee n)^k \varepsilon^{-2k}\delta^{-k}. \quad (C.14)$$
Since \( \eta k > 2, m/n \to \gamma \in (0, \infty) \), we know that for any \( \delta > 0 \),
\[
\sum_{n \in \mathbb{N}} \mathbb{P} \left( \frac{1}{(mn)^{1/4}} \| \Delta^{(1)} \|_{\text{op}} \geq \delta \right) < \infty. \tag{C.15}
\]

Borel Cantelli’s lemma implies for any \( \delta > 0 \),
\[
\lim_{n \to \infty, m/n \to \gamma} \frac{1}{(mn)^{1/4}} \| \Delta^{(1)} \|_{\text{op}} \leq \delta \quad \text{a.s..} \tag{C.16}
\]
Taking \( \delta \to 0 \), we get that,
\[
\limsup_{n \to \infty, m/n \to \gamma} \frac{1}{(mn)^{1/4}} \| \Delta^{(1)} \|_{\text{op}} = 0. \tag{C.17}
\]
Now, we take \( \varepsilon \to 0 \) on both sides of Eq (C.17) and get the desired claim of Eq (C.11) for \( j = 1 \).

Next, for any \( \varepsilon > 0 \), Eq (C.13) implies that,
\[
\lim_{n \to \infty, m/n \to \gamma} \frac{1}{(mn)^{1/2}} \mathbb{E} \| \Delta^{(1)} \|_{\text{op}}^2 = 0. \tag{C.18}
\]
By taking \( \varepsilon \to 0 \) on both sides of the equation, we get the desired claim of Eq (C.12) for \( j = 1 \).

Now, we show the key moment bound at Eq (C.13). To start with, we note that \( \mathbb{E} \Delta^{(1)} = 0 \).
Now, to upper bound \( \mathbb{E} \| \Delta^{(1)} \|_{\text{op}}^k \) we use the following lemma, whose proof is given in Section H.1.

**Lemma C.1.** For any \( k \geq 2 \), there exists some constant \( C_k > 0 \) depending solely on \( k \) such that for any matrix \( X \in \mathbb{R}^{m \times n} \) such that \( \mathbb{E} X = 0 \), and either (1) \( X \) has independent entries or (2) \( X \) is symmetric (in this case \( m = n \)) and has independent upper triangular entries,
\[
\mathbb{E} \| X \|_{\text{op}}^k \leq C_k (m + n)^k \left[ \max_{i \in [m]} \left( \mathbb{E} \sum_{j=1}^{n} X_{i,j}^2 \right)^{1/2} + \max_{j \in [n]} \left( \mathbb{E} \sum_{i=1}^{m} X_{i,j}^2 \right)^{1/2} + \mathbb{E} \max_{i,j} | X_{i,j} | \right]^k. \tag{C.19}
\]
A direct application of Lemma C.1 shows for any \( k \geq 2 \), there exists a constant \( C_k > 0 \), such that
\[
\mathbb{E} \| \Delta^{(1)} \|_{\text{op}}^k \leq C_k (m + n)^k \left[ \max_{i \in [m]} \left( \mathbb{E} \sum_{j=1}^{n} (\Delta^{(1)}_{i,j})^2 \right)^{1/2} + \max_{j \in [n]} \left( \mathbb{E} \sum_{i=1}^{m} (\Delta^{(1)}_{i,j})^2 \right)^{1/2} + \mathbb{E} \| \Delta^{(1)} \|_{\text{max}} \right]^k. \tag{C.20}
\]
To upper bound the RHS of Eq (C.20), we first show that, for some constant \( C, \varepsilon_0 \) independent of \( m, n \), we have, for all \( \varepsilon \leq \varepsilon_0 \),
\[
\| f_{W, \varepsilon} (\cdot) \|_{\infty} \leq C \varepsilon^{-2}. \tag{C.21}
\]
In fact, for all \( t \in \mathbb{R} \), we have,
\[
| f_{W, \varepsilon} (t) | = \left| \frac{p''_W (t)}{p_W (t)} + \frac{(p'_W (t))^2}{(p_W (t) + \varepsilon)^2} \right| \leq \varepsilon^{-2} \left( \varepsilon \| p''_W (\cdot) \|_{\infty} + \| p'_W (\cdot) \|_{\infty}^2 \right) \leq 2 \varepsilon^{-2} (M_2 \lor \varepsilon)^2,
\]
where the last inequality follows from Assumption A2. This gives the desired bound at Eq (C.21).

Now, by Eq (C.21), we know that, for \( \varepsilon \) small enough, the random variable \( f_{W, \varepsilon} (W_{1,1}) \) satisfies,
\[
\text{Var}^{1/2} (f_{W, \varepsilon} (W_{1,1})) \leq \| f_{W, \varepsilon} (\cdot) \|_{\infty} \leq C \varepsilon^{-2}. \tag{C.22}
\]
Hence we have the bound that,
\[
\max_{i \in [m]} \left( \mathbb{E} \left( \left( \Delta_{i,j}^{(1)} \right)^2 \right) \right)^{1/2} = \|X\|_{\ell_2 \to \ell_\infty} \times \text{Var}^{1/2} \left( f_{W,\varepsilon}^{\prime} (W_{1,1}) \right) \leq C\varepsilon^{-2}\|X\|_{\ell_2 \to \ell_\infty}. \tag{C.23}
\]

Similarly,
\[
\max_{j \in [n]} \mathbb{E} \left( \left( \Delta_{i,j}^{(1)} \right)^2 \right)^{1/2} = \|X^T\|_{\ell_2 \to \ell_\infty} \times \text{Var}^{1/2} \left( (f_{W,\varepsilon}^{\prime} (W_{1,1})) \right) \leq C\varepsilon^{-2}\|X^T\|_{\ell_2 \to \ell_\infty}. \tag{C.24}
\]

Lastly, since by Eq (C.21), we have \[\|\Delta^{(1)}\|_{\max} \leq C\varepsilon^{-2}\|X\|_{\max}, \tag{C.25}\]

Now, substituting Eq (C.23), Eq (C.24) and Eq (C.25) into Eq (C.20), we get for any \(k > 0\), there exists some constant \(C_k > 0\) such that

\[
\mathbb{E}\|\Delta^{(1)}\|^k_{\text{op}} \leq C_k \log(m + n)^k \varepsilon^{-2k} \max\{\|X\|_{\ell_2 \to \ell_\infty},\|X^T\|_{\ell_\infty \to \ell_2}\}^k \leq C_k (m \lor n)^{(1/2-\eta)k} \log(m \lor n)^k \varepsilon^{-2k},
\]

giving the desired claim at Eq (C.13).

**Proof of Eq (C.11) and Eq (C.12) for \(\Delta^{(2)}\).** In the proof, we show the following strengthened result: for some constant \(C\) independent of \(m, n\), we have for all \(m, n \in \mathbb{N}\) and \(\varepsilon > 0\),

\[
\|\Delta^{(2)}\|_{\text{op}} \leq C\varepsilon^{-3}(m \lor n)^{1/2-\eta}. \tag{C.26}
\]

Indeed, Eq (C.26) clearly implies for any \(\varepsilon > 0\),

\[
\lim_{n \to \infty, m/n \to \gamma} \frac{1}{(mn)^{1/4}} \|\Delta^{(2)}\|_{\text{op}} = 0. \tag{C.27}
\]

and

\[
\lim_{n \to \infty, m/n \to \gamma} \frac{1}{(mn)^{1/2}} \mathbb{E}\|\Delta^{(2)}\|^2_{\text{op}} = 0. \tag{C.28}
\]

Now taking limit on both sides of Eq (C.27) and Eq (C.28) shows that the desired claim at Eq (C.11) and Eq (C.12) hold for \(j = 2\).

In the rest of the proof, we show Eq (C.26). Indeed, we first show that, for some constant \(C, \varepsilon_0\) independent of \(m, n, \varepsilon\), we have, for \(\varepsilon \leq \varepsilon_0\),

\[
\|f_{W,\varepsilon}^{\prime\prime}(\cdot)\|_{\infty} \leq C\varepsilon^{-3}. \tag{C.29}
\]

In fact, for all \(t \in \mathbb{R}\),

\[
|f_{W,\varepsilon}^{\prime\prime}(t)| = \left| \frac{p_W^{\prime\prime}(t)}{p_W(t) + \varepsilon} - \frac{3p_W^{\prime\prime}(t)p_W^{\prime}(t)}{(p_W(t) + \varepsilon)^2} + \frac{2(p_W^{\prime}(t))^3}{(p_W(t) + \varepsilon)^3} \right| \leq\varepsilon^{-3} \left( |p_W^{\prime\prime}(\cdot)|_{\infty} \varepsilon^2 + 3 |p_W^{\prime}(\cdot)|_{\infty} |p_W^{\prime}(\cdot)|_{\infty} \varepsilon + 2 |p_W^{\prime}(\cdot)|_{\infty}^3 \right) \leq 6\varepsilon^{-3}(M_2 \lor \varepsilon)^3, \tag{C.30}
\]

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where the last inequality follows from Assumption A2. This proves the desired bound at Eq (C.29).

Now, fix the constant $C > 0$ such that Eq (C.29) holds. Introduce the auxiliary matrix $\Delta^{(2)}$ by

$$\Delta_{i,j}^{(2)} = \frac{1}{2} C \varepsilon^{-3} X_{i,j}^2.$$  
(C.31)

By construction of $\Delta$ and $\Delta^{(2)}$, we know that, there exists some $\varepsilon_0$ such that for any $\varepsilon \leq \varepsilon_0$,

$$\left| \Delta_{i,j}^{(2)} \right| \leq \frac{1}{2} X_{i,j}^2 \|f''_{W, \varepsilon}(\cdot)\|_\infty \leq \Delta_{i,j}^{(2)}.$$  
(C.32)

This immediately implies for any $\varepsilon \leq \varepsilon_0$, and any $l \in \mathbb{N}$,

$$\text{tr}\left\{ \left( \Delta^{(2)}(\Delta^{(2)^T})^l \right) \right\} \leq \text{tr}\left\{ \left( \Delta^{(2)}(\Delta^{(2)^T})^l \right) \right\}.$$  
(C.33)

Since $\|A\|_{op} = \lim_{l \to \infty} \text{tr}\{(A^T A)^l\}$ holds for any matrix $A$, we get that,

$$\|\Delta^{(2)}\|_{op} = \lim_{l \to \infty} \left( \text{tr}\left\{ \left( \Delta^{(2)}(\Delta^{(2)^T})^l \right) \right\} \right)^{\frac{1}{l}} \leq \lim_{l \to \infty} \left( \text{tr}\left\{ \left( \Delta^{(2)}(\Delta^{(2)^T})^l \right) \right\} \right)^{\frac{1}{l}} = \|\Delta^{(2)}\|_{op}.$$  
(C.34)

The above estimate immediately implies that

$$\|\Delta^{(2)}\|_{op} \leq \|\Delta^{(2)}\|_{op} = C \varepsilon^{-3} \|X \odot X\|_{op} \leq CM \varepsilon^{-3}(m \vee n)^{1/2-\eta},$$  
(C.35)

where the last inequality uses the assumption $X \in \mathcal{F}_{m,n}(r, M, \eta)$. This proves Eq (C.26) as desired.

**Proof of Eq (C.11) and Eq (C.12) for $\Delta^{(3)}$.** Now that $Z$ is a random matrix whose entries are i.i.d mean 0 and variance 1. Moreover, by Eq (C.7), we know for any fix $\varepsilon > 0$, there exists some constant $C_\varepsilon$ (independent of $m, n$) such that $\|Z\|_{\max} \leq C_\varepsilon$. Thus, random matrix theory show that, for any fix $\varepsilon > 0$, almost surely we have,

$$\lim_{n \to \infty, m/n \to \gamma} \frac{1}{(mn)^{1/4}} \|Z\|_{op} = \gamma^{1/4} + \gamma^{-1/4}.$$  
(C.36)

Moreover, for any fix $\varepsilon > 0$, we have

$$\lim_{n \to \infty, m/n \to \gamma} \frac{1}{(mn)^{1/2}} \mathbb{E}\|Z\|_{op}^2 = (\gamma^{1/4} + \gamma^{-1/4})^2.$$  
(C.37)

Since by Eq (C.6), we have,

$$\lim_{\varepsilon \to 0} \tilde{\nu}_\varepsilon = I_W^{1/2},$$  
(C.38)

Eq (C.36) and Eq (C.37) immediately imply that,

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty, m/n \to \gamma} \frac{1}{(mn)^{1/4}} \|\Delta^{(3)}\|_{op} = (\gamma^{1/4} + \gamma^{-1/4}) \cdot \lim_{\varepsilon \to 0} (\tilde{\nu}_\varepsilon - I_W^{1/2}) = 0,$$  
(C.39)

and moreover,

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty, m/n \to \gamma} \frac{1}{(mn)^{1/2}} \mathbb{E}\|\Delta^{(3)}\|_{op}^2 = (\gamma^{1/4} + \gamma^{-1/4})^2 \cdot \lim_{\varepsilon \to 0} (\tilde{\nu}_\varepsilon - I_W^{1/2})^2 = 0.$$  
(C.40)

This proves the desired claim of Eq (C.11) and Eq (C.12) for $\Delta^{(3)}$. 

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Proof of Eq (C.11) and Eq (C.12) for $\Delta^{(4)}$. By definition, $\|\Delta^{(4)}\|_{op} = |\mu'_\varepsilon - I_W| \|X\|_{op}$. Since $X \in F_{m,n}(r, M, \eta)$, we know that,

$$\lim_{n \to \infty, m/n \to \gamma} (mn)^{1/4} \frac{1}{\|\Delta^{(4)}\|_{op}} \leq |\mu'_\varepsilon - I_W| M.$$  \hspace{1cm} (C.41)

and

$$\lim_{n \to \infty, m/n \to \gamma} (mn)^{1/2} \mathbb{E} \|\Delta^{(4)}\|_{op}^2 \leq (\mu'_\varepsilon - I_W)^2 M^2.$$  \hspace{1cm} (C.42)

Since we have the expression below for $\mu'_\varepsilon$,

$$\mu'_\varepsilon = \int_{\mathbb{R}} f_{W,\varepsilon}(w)p_W(w) dw = -\int_{\mathbb{R}} f_{W,\varepsilon}(w)p'_W(w) dw = \int_{\mathbb{R}} \frac{(p'_W(w))^2}{p_W(w) + \varepsilon} dw,$$  \hspace{1cm} (C.43)

we know by dominated convergence theorem

$$\lim_{\varepsilon \to 0} |\mu'_\varepsilon - I_W| = 0.$$  \hspace{1cm} (C.44)

Hence, by taking $\varepsilon \to 0$ on both sides of Eq (C.41) and Eq (C.42), we prove the desired claim of Eq (C.11) and Eq (C.12) for $\Delta^{(4)}$.

D Proof of Lemma 7.4

Denote the quantities $W$, $I_{W,\varepsilon}$ and $\delta_{W,\varepsilon}$ to be

$$\mathcal{W} = \frac{1}{mn} \sum_{i \in [m], j \in [n]} W_{i,j}, \quad I_{W,\varepsilon} = \int_{\mathbb{R}} \frac{(p'_W(w))^2}{p_W(w) + \varepsilon} dw \quad \text{and} \quad \delta_{W,\varepsilon} = |I_{W,\varepsilon} - I_W|. \hspace{1cm} (D.1)$$

Denote $R_{n,1}$ and $R_{n,2}$ to be

$$R_{n,1} = h_n^2 + (mn h_n)^{-1/2} \log(mn) \quad \text{and} \quad R_{n,2} = (h'_n)^2 + (mn(h'_n)^3)^{-1/2} \log(mn). \hspace{1cm} (D.2)$$

Denote $R_n(\varepsilon)$ and $Q_n(\varepsilon; \kappa)$ to be

$$R_n(\varepsilon) = \varepsilon^{-1} R_{n,1} + \varepsilon^{-2} R_{n,2} \quad \text{and} \quad Q_n(\varepsilon; \kappa) = R_n(\varepsilon) + \varepsilon^{-2} (mn)^{-\kappa/2} + \delta_{W,\varepsilon} + \varepsilon. \hspace{1cm} (D.3)$$

The crux of the proof is to provide high probability bounds onto the quantities

$$\|\hat{f}_{Y,\varepsilon}(Y) - f_{W,\varepsilon}(Y)\|_{op} \quad \text{and} \quad |I_{W,\varepsilon} - I_W|. \hspace{1cm} (D.4)$$

The bounds are made more precise in the technical lemma below. As the proof is lengthy, we defer its proof into Section D.1.

Lemma D.1. Assume that $h_n, h'_n$ are chosen in a way such that,

$$\lim_{n} R_{n,1} = \lim_{n} R_{n,2} = 0. \hspace{1cm} (D.5)$$
Fix $\kappa \in (0, 1/2)$. There exist some constants $C, c > 0$ independent of $m, n, \varepsilon$ (but can be dependent of $\kappa$ and underlying distribution $\mathbb{P}$), such that whenever $\varepsilon \leq c$, there exists $n_0 = n_0(\varepsilon)$ such that, for any $n \geq n_0$, we have, with probability at least $1 - (mn)^{-1 - \kappa}$,
\[ \left\| \hat{f}_{Y, \varepsilon}(Y) - f_{W, \varepsilon}(Y) \right\|_{\text{op}} \leq C (mn)^{1/2} (R_n(\varepsilon) + \varepsilon^{-2} |W|). \] (D.6)
and
\[ \left| \hat{I}_{W, \varepsilon} - I_W \right| \leq C Q_n(\varepsilon; \kappa) \] (D.7)

Now, we prove Eq (7.20), Eq (7.21) and Eq (7.22) in the three paragraphs below. This gives the desired result of Lemma 7.4.

**Proof of Eq (7.20)** Let $\kappa = 1/4$. According to Lemma D.1, we know that, there exist some constants $c, C > 0$, such that for any $\varepsilon \leq c$, the event $\Lambda_n(\varepsilon)$ defined by
\[ \Lambda_n(\varepsilon) = \left\{ \left\| \hat{f}_{Y, \varepsilon}(Y) - f_{W, \varepsilon}(Y) \right\|_{\text{op}} \geq C (mn)^{1/4} (R_n(\varepsilon) + \varepsilon^{-2} |W|) \right\} \] (D.8)
happens with probability at most $(mn)^{-3/4}$ for large enough $n$. As $m/n \to \gamma$, this shows that,
\[ \sum_{n=1}^{\infty} \mathbb{P}(\Lambda_n(\varepsilon)) < \infty. \] (D.9)

Applying Borel Cantelli’s lemma, the above shows that, for any $\varepsilon \leq c$, the events $\{ \Lambda_n(\varepsilon) \}_{n=1}^{\infty}$ happen at most finite times almost surely. Thus, for $\varepsilon \leq c$, we know that almost surely
\[ \limsup_{n \to \infty, m/n \to \gamma} \left( \frac{1}{(mn)^{1/4}} \left\| \hat{f}_{Y, \varepsilon}(Y) - f_{W, \varepsilon}(Y) \right\|_{\text{op}} - C (mn)^{1/4} (R_n(\varepsilon) + \varepsilon^{-2} |W|) \right) \leq 0. \] (D.10)

Now that, since $\{W_{i,j}\}_{i \in [m], j \in [n]}$ are i.i.d mean 0 with finite second moments, we know by the law of the iterated logarithm that,
\[ \limsup_{n \to \infty, m/n \to \gamma} (mn)^{1/4} |W| \overset{a.s.}{\to} 0. \] (D.11)

Moreover, our choice of $h_n = n^{-m}$ and $h'_n = n^{-\eta_2}$ for $\eta_1 \in (1/4, 1)$ and $\eta_2 \in (1/4, 1/3)$ gives
\[ \limsup_{n \to \infty, m/n \to \gamma} (mn)^{1/4} \max\{R_{n,1}, R_{n,2}\} \to 0 \] (D.12)
which by definition of $R_n(\varepsilon)$, implies that
\[ \limsup_{n \to \infty, m/n \to \gamma} (mn)^{1/4} R_n(\varepsilon) = 0 \] (D.13)

Hence, Eq (D.10), Eq (D.11) and Eq (D.13) together imply that, for $\varepsilon \leq c$, almost surely,
\[ \limsup_{n \to \infty, m/n \to \gamma} \left( \frac{1}{(mn)^{1/4}} \left\| \hat{f}_{Y, \varepsilon}(Y) - f_{W, \varepsilon}(Y) \right\|_{\text{op}} \right) = 0. \] (D.14)

By taking $\varepsilon \to 0$, we derive the desired claim at Eq (7.20).
**Proof of Eq (7.21)** In the proof, we set $\kappa = \kappa_0$ where $\kappa_0 \in (0, 1/24)$. We show that Eq (7.21) holds for $\nu_0 = \nu_0$ for any $\nu_0 \in (0, 1/24)$. To start with, by Lemma D.1, we know that, for some constants $c, C > 0$ independent of $m, n, \varepsilon$, we have for any $\varepsilon \leq c$, the event $\Lambda_n(\varepsilon; \kappa_0)$ defined by

$$
\Lambda_n(\varepsilon; \kappa_0) = \left\{ \left\| \hat{f}_Y, \varepsilon (Y) - f_{W, \varepsilon} (Y) \right\|_{\text{op}} \geq C (mn)^{1/2} (R_n(\varepsilon) + \varepsilon^{-2} \|W\|) \right\}
$$

(D.15)

happens with probability at least $(mn)^{-(1-\kappa_0)}$. To prove the desired Eq (7.21), it suffices to show

$$
\lim_{\varepsilon \to 0} \limsup_{n \to \infty, m/n \to \gamma} \frac{1}{(mn)^{(1+\nu_0)/4}} \mathbb{E} \left[ \left\| \hat{f}_Y, \varepsilon (Y) - f_{W, \varepsilon} (Y) \right\|_{\text{op}}^{(1+\nu_0)} 1 \{ \Lambda_n(\varepsilon; \kappa_0)^C \} \right] = 0.
$$

(D.16)

and

$$
\lim_{\varepsilon \to 0} \limsup_{n \to \infty, m/n \to \gamma} \frac{1}{(mn)^{(1+\nu_0)/4}} \mathbb{E} \left[ \left\| \hat{f}_Y, \varepsilon (Y) - f_{W, \varepsilon} (Y) \right\|_{\text{op}}^{(1+\nu_0)} 1 \{ \Lambda_n(\varepsilon; \kappa_0) \} \right] = 0.
$$

(D.17)

First, we show Eq (D.16). Note that, Eq (D.13) shows that under our careful choice of $h_n = n^{-\eta_1}$ and $h'_n = n^{-\eta_2}$ for $\eta_1 \in (1/4, 1)$ and $\eta_2 \in (1/4, 1/3)$, we have

$$
\lim_{\varepsilon \to 0} \lim_{n \to \infty, m/n \to \gamma} (mn)^{1/4} (R_n(\varepsilon))^2 = 0.
$$

(D.18)

Next, by Assumption A2, $\{W_{i,j}\}_{i \in [m], j \in [n]}$ are i.i.d mean 0 with bounded second moments. Thus,

$$
\lim_{n \to \infty, m/n \to \gamma} \mathbb{E}( (mn)^{1/4} \|W\|)^2 = 0,
$$

(D.19)

which by taking $\varepsilon \to 0$, immediately implies that,

$$
\lim_{\varepsilon \to 0} \lim_{n \to \infty, m/n \to \gamma} \mathbb{E} \left[ \left( (mn)^{1/4} \varepsilon^{-2} \|W\| \right)^2 \right] = 0.
$$

(D.20)

Now, Eq (D.18), Eq (D.20) and Hölder’s inequality immediately give that

$$
\lim_{\varepsilon \to 0} \limsup_{n \to \infty, m/n \to \gamma} \mathbb{E} \left[ \left( (mn)^{1/4} (R_n(\varepsilon) + \varepsilon^{-2} \|W\|) \right)^2 \right] = 0.
$$

(D.21)

Finally by definition of $\Lambda_n(\varepsilon)$, the above equation implies that,

$$
\lim_{\varepsilon \to 0} \limsup_{n \to \infty, m/n \to \gamma} \frac{1}{(mn)^{1/2}} \mathbb{E} \left[ \left\| \hat{f}_Y, \varepsilon (Y) - f_{W, \varepsilon} (Y) \right\|_{\text{op}}^2 1 \{ \Lambda_n(\varepsilon; \kappa_0)^C \} \right] = 0,
$$

(D.22)

which by Hölder’s inequality again, shows that Eq (D.16) holds for any $\nu_0 \in (0, 1/24)$.

Next, we prove Eq (D.17). The starting point of the proof is the following bound on the operator norm (see Lemma 1.3 for details),

$$
\frac{1}{(mn)^{1/2}} \left\| \hat{f}_Y, \varepsilon (Y) - f_{W, \varepsilon} (Y) \right\|_{\text{op}} \leq \left\| \hat{f}_Y, \varepsilon (Y) - f_{W, \varepsilon} (Y) \right\|_{\max} \leq \left\| \hat{f}_Y, \varepsilon (Y) \right\|_{\max} + \left\| f_{W, \varepsilon} (Y) \right\|_{\max}.
$$

(D.23)
To upper bound the RHS of Eq (D.23), we first note that, almost surely,
\[ \left\| \hat{f}_{Y,\varepsilon}(Y) \right\|_{\max} \leq \left\| \hat{f}_{Y,\varepsilon}(\cdot) \right\|_{\infty} \leq \varepsilon^{-1} \left\| p_{W}(\cdot) \right\|_{\infty} \leq \varepsilon^{-1} (h'_{n})^{-2} \left\| K'(\cdot) \right\|_{\infty} \leq M\varepsilon^{-1} (h'_{n})^{-2}. \] (D.24)
Moreover, by Assumption A2, we have,
\[ \left\| f_{W,\varepsilon}(Y) \right\|_{\max} \leq \left\| f_{W,\varepsilon}(\cdot) \right\|_{\infty} \leq \varepsilon^{-1} \left\| p'_{W}(\cdot) \right\|_{\infty} \leq M_{2} \varepsilon^{-1}. \] (D.25)
Substituting Eq (D.24) and Eq (D.25) into Eq (D.23), we get for some constant C independent of m, n, \varepsilon, the estimate below holds almost surely,
\[ \frac{1}{(mn)^{1/4}} \left\| \hat{f}_{Y,\varepsilon}(Y) - f_{W,\varepsilon}(Y) \right\|_{\op} \leq C(mn)^{1/4} \varepsilon^{-1} ((h'_{n})^{-2} + 1). \] (D.26)
Hence, this implies
\[ \frac{1}{(mn)^{1/4}} \mathbb{E} \left[ \left\| \hat{f}_{Y,\varepsilon}(Y) - f_{W,\varepsilon}(Y) \right\|_{\op}^{1+\nu_{0}} \cdot 1 \{ \Lambda_{n}(\varepsilon; \kappa_{0}) \} \right] \leq C^{2} (mn)^{(1+\nu_{0})/4} \varepsilon^{-(1+\nu_{0})} (h'_{n})^{-2} + 1 \cdot (1+\nu_{0}) \mathbb{P}(\Lambda_{n}(\varepsilon; \kappa_{0})) \leq C^{2} \varepsilon^{-(1+\nu_{0})} (mn)^{-3/4 - \kappa_{0} - \nu_{0}/4} ((h'_{n})^{-2} + 1)^{(1+\nu_{0})}. \] (D.27)
Since \( h'_{n} = n^{-\eta_{2}} \) for \( \eta_{2} \in (1/4, 1/3) \), we know that for our choice of \( \nu_{0}, \kappa_{0} \in (0, 1/24) \),
\[ \lim_{n \to \infty, m/n \to \gamma} (mn)^{-3/4 - \kappa_{0} - \nu_{0}/4} ((h'_{n})^{-2} + 1)^{(1+\nu_{0})} = 0 \] (D.28)
holds for any \( \varepsilon > 0 \). Thus Eq (D.28) shows that for the \( \kappa_{0}, \nu_{0} > 0 \),
\[ \lim_{n \to \infty, m/n \to \gamma} \frac{1}{(mn)^{1/4}} \mathbb{E} \left[ \left\| \hat{f}_{Y,\varepsilon}(Y) - f_{W,\varepsilon}(Y) \right\|_{\op}^{1+\nu_{0}} \cdot 1 \{ \Lambda_{n}(\varepsilon; \kappa_{0}) \} \right] = 0 \] (D.29)
holds for any \( \varepsilon > 0 \). Taking \( \varepsilon \to 0 \) gives the desired Eq (D.17).

**Proof of Eq (7.22)** In the proof, we fix \( \kappa = 1/4 \). To simplify our notation, denote,
\[ \bar{Q}_{n}(\varepsilon) = Q_{n}(\varepsilon; 1/4) = R_{n}(\varepsilon) + \varepsilon^{-2} (mn)^{-1/8} + \delta_{W,\varepsilon} + \varepsilon. \] (D.30)
According to Lemma D.1, we know that, there exist some constants \( c, C > 0 \), such that for any \( \varepsilon \leq c \), the event \( \Lambda_{n}(\varepsilon) \) defined by
\[ \Lambda_{n}(\varepsilon) = \left\{ |\hat{I}_{W,\varepsilon} - I_{W}| \geq C \bar{Q}_{n}(\varepsilon) \right\} \] (D.31)
leads with probability at most \( (mn)^{-3/4} \) for large enough \( n \). As \( m/n \to \gamma \), this shows that,
\[ \sum_{n=1}^{\infty} \mathbb{P}(\Lambda_{n}(\varepsilon)) < \infty. \] (D.32)
Applying Borel Cantelli’s lemma, the above shows that, for any \( \varepsilon \leq c \), the events \( \{ \Lambda_{n}^{I}(\varepsilon) \}_{n=1}^{\infty} \) happen at most finite times almost surely. Thus, for \( \varepsilon \leq c \), almost surely
\[
\limsup_{n \to \infty, m/n \to \gamma} |\hat{I}_{W,\varepsilon} - I_{W}| \leq C \limsup_{n \to \infty, m/n \to \gamma} Q_{n}(\varepsilon) \leq C(\delta_{W,\varepsilon} + \varepsilon). \tag{D.34}
\]
By taking \( \varepsilon \to 0 \), we know that
\[
\lim_{\varepsilon \to 0} \limsup_{n \to \infty, m/n \to \gamma} |\hat{I}_{W,\varepsilon} - I_{W}| \leq C \lim_{\varepsilon \to 0}(\delta_{W,\varepsilon} + \varepsilon) = 0, \tag{D.35}
\]
which gives the desired claim of Eq (7.22).

D.1 Proof of Lemma D.1

D.1.1 Notation

Throughout the section, we make use of the following notation. We use constants \( C, c \) to denote constants that are independent of \( m, n, \varepsilon \), but can be dependent on the underlying distribution \( P \), the model parameters \( M, M_{1}, M_{2}, r, \eta \) and \( \kappa \). We also use \( C, c \) to denote numerical constants in some situations, where we shall point this fact explicitly. It is understood that all those constants \( C, c \) might not be the same at each occurrence. For any matrix \( A \in \mathbb{R}^{m \times n} \), we denote
\[
\bar{A} = \frac{1}{mn} \sum_{i \in [m], j \in [n]} A_{i,j}, \quad \|A\|_{\max} = \max_{i \in [m], j \in [n]} |A_{i,j}| \quad \text{and} \quad \tilde{A} = A - \bar{A}1_{m}1_{n}^{T}. \tag{D.36}
\]

D.1.2 Proof

The crux of the proof is the key Lemma D.2 and its companion Lemma D.3, which we present below. The proof of the two lemma are deferred to Section D.1.3.

**Lemma D.2.** For some constant \( C \), we have with probability at least \( 1 - (mn)^{-(1-\kappa)} \),
\[
\|\hat{p}_{Y}(Y) - p_{W}(Y)\|_{\max} \leq C (R_{n,1} + |W|). \tag{D.37}
\]
Moreover, for the same constant \( C \), we have with probability at least \( 1 - (mn)^{-(1-\kappa)} \),
\[
\|\hat{p}_{Y}'(Y) - p_{W}'(Y)\|_{\max} \leq C (R_{n,2} + |W|). \tag{D.38}
\]

**Lemma D.3.** For some constant \( C \), we have with probability at least \( 1 - (mn)^{-(1-\kappa)} \),
\[
\|\hat{p}_{Y}(\tilde{Y}) - p_{W}(\tilde{Y})\|_{\max} \leq C (R_{n,1} + |W|), \tag{D.39}
\]
Moreover, for the same constant \( C \), we have with probability at least \( 1 - (mn)^{-(1-\kappa)} \),
\[
\|\hat{p}_{Y}'(\tilde{Y}) - p_{W}'(\tilde{Y})\|_{\max} \leq C (R_{n,2} + |W|). \tag{D.40}
\]

Given Lemma D.2 and Lemma D.3, we show the desired Eq (D.6) and Eq (D.7).
Proof of Eq (D.6) Our proof starts from the following upper bound on the operator norm (see Lemma I.3 for a proof of this bound)
\[
\left\| \hat{f}_{Y,\varepsilon}(Y) - f_{W,\varepsilon}(Y) \right\|_{op} \leq (mn)^{1/2} \left\| \hat{f}_{Y,\varepsilon}(Y) - f_{W,\varepsilon}(Y) \right\|_{max}.
\]
According to Eq (D.41), it suffices to show that, with probability at least \(1 - (mn)^{-(1-\kappa)}\):
\[
\left\| \hat{f}_{Y,\varepsilon}(Y) - f_{W,\varepsilon}(Y) \right\|_{max} \leq C(R_n(\varepsilon) + \varepsilon^{-2}|W|).
\]
For notational simplicity, denote the function \(g : \mathbb{R} \times \mathbb{R}_+ \rightarrow \mathbb{R}\) to be:
\[
g(s, t) = -\frac{s}{t + \varepsilon} \text{ for any } s \in \mathbb{R}, \ t \in \mathbb{R}_+.
\]
Using this notation, the goal is equivalent to showing that, with probability at least \(1 - (mn)^{-(1-\kappa)}\):
\[
\left\| g(\hat{p}'_Y(Y), \hat{p}_Y(Y)) - g(p'_W(Y), p_W(Y)) \right\|_{max} \leq C(R_n(\varepsilon) + \varepsilon^{-2}|W|).
\]
Motivated by Eq (D.44), we present an elementary lemma regarding the function \(g(\cdot, \cdot)\), basically showing that the function \(g\) is (pseudo)-Lipschitz with respect to its input arguments.

Lemma D.4. For any \(s_1, s_2 \in \mathbb{R}\) and, \(t_1, t_2 \geq 0,\)
\[
|g(s_1, t_1) - g(s_2, t_2)| \leq \varepsilon^{-1}|s_1 - s_2| + \varepsilon^{-2}|t_1 - t_2|((|s_1| \wedge |s_2|)).
\]

Lemma D.4 immediately gives us a control on the LHS of Eq (D.44). In fact, for \(i \in [m], j \in [n],\)
\[
\begin{align*}
\hat{f}_{Y,\varepsilon}(Y_{i,j}) - f_{W,\varepsilon}(Y_{i,j}) &= |g(p'_W(Y_{i,j}), p_W(Y_{i,j})) - g(p'_W(Y_{i,j}), p_W(Y_{i,j}))| \\
&\leq \varepsilon^{-1}|\hat{p}'_Y(Y_{i,j}) - p'_W(Y_{i,j})| + \varepsilon^{-2}|\hat{p}_Y(Y_{i,j}) - p_W(Y_{i,j})||(\hat{p}'_Y(Y_{i,j}) \wedge p'_W(Y_{i,j}))| \\
&= \varepsilon^{-1}|\hat{p}'_Y(Y) - p'_W(Y)||_{max} + \varepsilon^{-2}\|p'_W(\cdot)\|_{\infty} \|\hat{p}_Y(Y) - p_W(Y)\|_{max}.
\end{align*}
\]
Since \(\|p'_W(\cdot)\|_{\infty} \leq M_2\) by Assumption A2, the desired claim of Eq (D.6) follows from the above estimate and Lemma D.2.

Proof of Eq (D.7) Recall the definition of \(\delta_{W,\varepsilon}\) at Eq (D.1). Our proof starts from the estimate
\[
|\hat{I}_{W,\varepsilon} - I_W| \leq |\hat{I}_{W,\varepsilon} - I_{W,\varepsilon}| + \delta_{W,\varepsilon}.
\]
Recall the definition of function \(g : \mathbb{R}^2 \rightarrow \mathbb{R}\) in Eq (D.43). We can express the estimator \(\hat{I}_{W,\varepsilon}\) as,
\[
\hat{I}_{W,\varepsilon} = I_{W,\varepsilon} + \varepsilon
\]
where we define
\[
\hat{I}_{W,\varepsilon} := \frac{1}{mn} \sum_{i \in [m], j \in [n]} g^2(\hat{p}'_Y(Y_{i,j}), \hat{p}_Y(Y_{i,j})).
\]
Now, define the auxiliary random variable \(\hat{I}_{W,\varepsilon}\) to be:
\[
\hat{I}_{W,\varepsilon} = \frac{1}{mn} \sum_{i \in [m], j \in [n]} g^2(p'_W(W_{i,j}), p_W(W_{i,j})).
\]
By triangle inequality and Eq (D.45), we can upper bound the difference between \( \hat{I}_{W,\varepsilon} \) and \( I_W \) by

\[
\left| \hat{I}_{W,\varepsilon} - I_W \right| \leq \varepsilon_1 + \varepsilon_2 + (\delta_{W,\varepsilon} + \varepsilon) \quad \text{for} \quad \varepsilon_1 := \left| \hat{I}_{W,\varepsilon} - I_{W,\varepsilon} \right| \quad \text{and} \quad \varepsilon_2 := \left| I_{W,\varepsilon} - \bar{I}_{W,\varepsilon} \right|. \tag{D.49}
\]

We upper bound the error \( \varepsilon_1 \) first. Indeed, the next lemma provides a high probability onto \( \varepsilon_1 \), whose proof is deferred in Appendix Section H.3.

**Lemma D.5.** The following inequality holds for all \( t > 0 \),

\[
P(\varepsilon_1 > t + \delta_{W,\varepsilon}) \leq 2 \exp \left( - \frac{mnt^2 \varepsilon_4^4}{2M_2^4} \right). \tag{D.50}
\]

Now, if we plug in \( t = 2\varepsilon^{-2}M_2^2(mn)^{-1/2} \log(mn) \) into Eq (D.50), we know that, for some constant \( C > 0 \), we have with probability at least \( 1 - (mn)^{-2(1-\kappa)} \),

\[
\varepsilon_1 \leq C\varepsilon^{-2}(mn)^{-1/2} \log(mn) + \delta_{W,\varepsilon}. \tag{D.51}
\]

Next, we upper bound error \( \varepsilon_2 \). We introduce the auxiliary quantities \( T_{n,1} \) and \( T_{n,2} \) below,

\[
T_{n,1}^2 = \left\| \hat{\rho}_Y(\bar{Y}) - p_W(\bar{Y}) \right\|_{\max}^2 + \frac{1}{mn} \| X \|_F^2 + \bar{W}^2, \tag{D.52}
\]

\[
T_{n,2}^2 = \left\| \hat{\rho}'_Y(\bar{Y}) - p'_W(\bar{Y}) \right\|_{\max}^2 + \frac{1}{mn} \| X \|_F^2 + \bar{W}^2. \tag{D.53}
\]

The following Lemma D.6 provides a deterministic upper bound on the error term \( \varepsilon_2 \). The proof is based on tedious calculations, and is deferred into appendix Section H.4.

**Lemma D.6.** There exists some constant \( C \) depending on \( M, M_1, M_2 \) such that,

\[
\varepsilon_2 \leq C(\varepsilon^{-2}T_{n,1} + \varepsilon^{-1}T_{n,2}) \left( \bar{I}_{W,\varepsilon}^{1/2} + \varepsilon^{-2}T_{n,1} + \varepsilon^{-1}T_{n,2} \right). \tag{D.54}
\]

Motivated by Lemma D.6, we study high probability upper bounds onto \( \bar{I}_{W,\varepsilon} \), \( T_{n,1} \) and \( T_{n,2} \). We start by giving high probability bound on \( \bar{I}_{W,\varepsilon} \). Indeed, by triangle inequality and Eq (D.51), we know with probability at least \( 1 - (mn)^{-2(1-\kappa)} \),

\[
\bar{I}_{W,\varepsilon} \leq C\varepsilon^{-2}(mn)^{-1/2} \log(mn) + I_{W,\varepsilon} + \delta_{W,\varepsilon}. \tag{D.55}
\]

Now that by definition \( I_{W,\varepsilon} \leq I_W \) and thus by triangle inequality \( \delta_{W,\varepsilon} \leq 2I_W \). This shows that, for some constant \( C > 0 \), with probability at least \( 1 - (mn)^{-2(1-\kappa)} \),

\[
\bar{I}_{W,\varepsilon} \leq C\varepsilon^{-2}(mn)^{-1/2} \log(mn) + 3I_W. \tag{D.56}
\]

Next, we give high probability upper bounds on \( T_{n,1} \) and \( T_{n,2} \). To do so, we note the lemma below. The proof is given in appendix Section H.5.

**Lemma D.7.** The following inequality holds for all \( t > 0 \),

\[
P\left( |\bar{W}| \geq t(mn)^{-1/2} \right) \leq t^{-2}M_1.
\]
Now, according to Lemma D.7, \(|\hat{W}| \leq M^{1/2}(mn)^{-\kappa/2}\) with probability at least \(1 - (mn)^{-(1-\kappa)}\). Since by assumption \(X \in F_{m,n}(r, M, \eta)\), we have
\[
\|X\|_F \leq \text{rank}^{1/2}(X) \|X\|_{op} \leq r^{1/2}M(mn)^{1/4}. \tag{D.57}
\]
Lemma D.3 now implies that with probability at least \(1 - (mn)^{-(1-\kappa)}\),
\[
T_{n,1} \leq C(R_{n,1} + (mn)^{-\kappa/2}) \quad \text{and} \quad T_{n,2} \leq C(R_{n,2} + (mn)^{-\kappa/2}). \tag{D.58}
\]
Plugging Eq (D.56) and Eq (D.58) into Eq (D.54), we get with probability at least \(1 - (mn)^{-(1-\kappa)}\)
\[
\mathcal{E}_2 \leq C \left(R_n(\varepsilon) + \varepsilon^{-2}(mn)^{-\kappa/2}\right) \left(R_n(\varepsilon) + \varepsilon^{-2}(mn)^{-\kappa/2} + \varepsilon^{-1}(mn)^{-1/4} \log(mn)^{1/2} + 1^{1/2}_{W}\right). \tag{D.59}
\]
Now for some constant \(C, c > 0\), we know when \(\varepsilon \leq c\), there exists \(n_0 = n_0(\varepsilon)\) such that for \(n \geq n_0\),
\[
R_n(\varepsilon) + \varepsilon^{-2}(mn)^{-\kappa/2} + \varepsilon^{-1}(mn)^{-1/4} \log(mn)^{1/2} + 1^{1/2}_{W} \leq C,
\]
and hence by Eq (D.59),
\[
\mathcal{E}_2 \leq C \left(R_n(\varepsilon) + \varepsilon^{-2}(mn)^{-\kappa/2}\right) \tag{D.60}
\]
The desired result in Eq (D.7) now follows by plugging Eq (D.59) and Eq (D.51) into Eq (D.49).

### D.1.3 Proof of Lemma D.2 and Lemma D.3

We only prove the high probability result for Eq (D.37) and Eq (D.39) since we can show the similar results for Eq (D.38) and Eq (D.40) in a similar way. For any matrix \(A \in \mathbb{R}^{m \times n}\), denote the following auxiliary function
\[
\hat{p}_W(x; A) = \frac{1}{mnh_n} \sum_{i \in [m], j \in [n]} K \left(\frac{W_{i,j} + A_{i,j} - A - x}{h_n}\right) \quad \text{and} \quad \hat{q}_W(x; A) = \mathbb{E}\hat{p}_W(x; A), \tag{D.61}
\]
where the expectation in definition of \(\hat{q}_W\) is taken with respect to the random matrix \(\hat{W}\). There is a connection between \(\hat{p}_W\) and \(\hat{p}_Y\): we have \(\hat{p}_Y(x) = \hat{p}_W(x + \hat{W}; X)\) for \(x \in \mathbb{R}\). Now, using those notation, our target is to show that, for some constant \(C\), with probability at least \(1 - (mn)^{-(1-\kappa)}\),
\[
\left\|\hat{p}_W(Y + \hat{W}1_m1_n^T; X) - p_W(Y)\right\|_{\max} \leq C \left(R_{n,1} + |\hat{W}|\right), \tag{D.62}
\]
and with probability at least \(1 - (mn)^{-(1-\kappa)}\),
\[
\left\|\hat{p}_W(Y + \hat{W}1_m1_n^T; X) - p_W(Y)\right\|_{\max} \leq C \left(R_{n,1} + |\hat{W}|\right). \tag{D.63}
\]
The proof of Eq (D.62) and Eq (D.63) is based on standard arguments in empirical process theory. It is convenient to list our proof strategies into the following three steps.

1. In the first step, we show that, with high probability the magnitude of \(\hat{W}\) is ‘small’ and that of \(Y\) is not ‘too large’. More precisely, for \(\eta_n = M_1^{1/2}(mn)^{-\kappa/2}\) and \(T_n = 2M_1^{1/2}(mn)\), with probability at least \(1 - (mn)^{-(1-\kappa)}\),
\[
|\hat{W}| \leq \eta_n, \; \|Y\|_{\max} \leq T_n \quad \text{and} \quad \|\hat{Y}\|_{\max} \leq T_n. \tag{D.64}
\]
Note that \(T_n \geq \eta_n\) from our definition.
2. In the second step, we show that, with high probability \( \hat{p}_W(x; X) \) is 'close' to \( \hat{q}_W(x; X) \) on the interval \( x \in [-2T_n, 2T_n] \). More precisely, we show there exists a numerical constant \( c > 0 \), such that if we denote \( Q_n = 4(mn)^{1/2}h_{n}^{-3/2}T_n \), then for any \( t > 0 \),

\[
\sup_{x \in [-2T_n, 2T_n]} |\hat{p}_W(x; X) - \hat{q}_W(x; X)| \leq 3(mnh_n)^{-1/2}t \tag{D.65}
\]

holds with probability at least \( 1 - MQ_n t^{-1} \exp(-ct^2/(M(t + M_2))) \).

3. In the last step, we prove a uniform upper bound on the difference between \( \hat{q}_W(x + W; X) \) to \( p_W(x) \). More precisely, we show that, with probability one,

\[
\sup_{x \in \mathbb{R}} |\hat{q}_W(x; X) - p_W(x - W)| \leq M_2 \left( Mh_n^2 + \frac{1}{mn} \|X\|_F^2 + |W| \right) . \tag{D.66}
\]

Now, we prove the desired target in Eq (D.62) using Eq (D.64), Eq (D.65) and Eq (D.66). Indeed, for some numerical constant \( C \) that is sufficiently large if we plug \( t = C(M \vee M_2) \log(mn)/3 \) into Eq (D.65), we get with probability at least \( 1 - (mn)^{-(1-\kappa)} \),

\[
\sup_{x \in [-2T_n, 2T_n]} |\hat{p}_W(x; X) - \hat{q}_W(x; X)| \leq C(M \vee M_2)(mnh_n)^{-1/2} \log(mn) . \tag{D.67}
\]

Thus, by triangle inequality, we know that, with probability at least \( 1 - (mn)^{-(1-\kappa)} \),

\[
\|\hat{p}_W(Y + W1_m 1_n^T; X) - p_W(Y)\|_{\text{max}} \leq C(M \vee M_2) \left( Mh_n^2 + \frac{1}{mn} \|X\|_F^2 + |W| \right) . \tag{D.68}
\]

Eq (D.62) now follows by Eq (D.57). Note that, Eq (D.63) can be proven in a similar way.

**Proof of Eq (D.64)** The following lemma gives high probability upper bound on \( \|W\|_{\text{max}} \). It is proven in appendix Section H.5.

**Lemma D.8.** The following inequality holds for all \( t > 0 \),

\[
\mathbb{P} \left( \|W\|_{\text{max}} \geq tM_1^{1/2}(mn)^{1/2} \right) \leq t^{-2} .
\]

Now, according to triangle inequality, we have,

\[
\|Y\|_{\text{max}} \leq \|W\|_{\text{max}} + \|X\|_{\text{max}} \leq \|W\|_{\text{max}} + \|X\|_F ,
\]

and

\[
\|\tilde{Y}\|_{\text{max}} \leq \|\tilde{W}\|_{\text{max}} + \|\tilde{X}\|_{\text{max}} \leq \|W\|_{\text{max}} + |W| + \|X\|_F .
\]

where in the last step we use \( \|\tilde{X}\|_{\text{max}} \leq \|\tilde{X}\|_F \leq \|X\|_F \). Now, Eq (D.64) follows from the high probability bound in Lemma D.7, Lemma D.8 and Eq (D.57).
Proof of Eq (D.65) The proof is based on standard uniform convergence arguments. We use \( I_n \) to denote the interval \( I_n = [-2T_n, 2T_n] \). Our proof proceeds in three steps.

1. First, there exists some numerical constant \( c > 0 \), such that for any fix \( x \in \mathbb{R} \) and \( t > 0 \)
\[
P \left( |\hat{p}_W(x; \mathbf{X}) - \hat{q}_W(x; \mathbf{X})| \geq (mh_n)^{-1/2}t \right) \leq 2 \exp \left( - \frac{ct^2}{M(t + M_2)} \right). \tag{D.69}
\]

2. Next, we show both \( \hat{p}_W(\cdot; \mathbf{X}) \) and \( \hat{q}_W(\cdot; \mathbf{X}) \) are Lipschitz functions with Lipschitz constant \( L_n = Mh_n^{-2} \).

3. Lastly, we use the covering type argument to show the uniform convergence result in Eq (D.65).

The proof of our first step follows by a more general result (which we state as Lemma D.9), whose proof is deferred into Section II.7.

**Lemma D.9.** Let \( \{X_i\}_{i=1}^n \) be independent continuous random variables with densities \( \{p_{X_i}\}_{i=1}^n \). Let \( K(\cdot) \) be square integrable on \( \mathbb{R} \). Denote \( \sigma^2, p_\infty, M_\infty \) to be the following quantities:
\[
p_\infty = \max_{i \in [n]} \|p_{X_i}(\cdot)\|_\infty, \quad M_\infty = \|K(\cdot)\|_\infty \quad \text{and} \quad \sigma^2 = \int_\mathbb{R} K^2(z) \, dz,
\]
where \( p_{X_i} \) is the density function of \( X_i \). For some \( h > 0 \), consider the following function
\[
Z_n(x) = \frac{1}{nh} \sum_{i=1}^n K_{h,X_i}(x) \quad \text{where} \quad K_{h,X_i}(x) := K \left( \frac{x - X_i}{h} \right).
\]
Assume \( nh \geq 1 \). Then, for some numerical constant \( c > 0 \), we have for all \( x \in \mathbb{R} \) and \( t > 0 \),
\[
P \left( |Z_n(x) - EZ_n(x)| \geq (nh)^{-1/2}t \right) \leq 2 \exp \left( - \frac{ct^2}{\sigma^2 p_\infty + M_\infty t} \right). \tag{D.70}
\]
To be precise, Eq (D.69) follows by plugging \( p_\infty = M_2, M_\infty = \sigma^2 = M \) into Eq (D.70).

Next, we show that both \( \hat{p}_W(\cdot; \mathbf{X}) \) and \( \hat{q}_W(\cdot; \mathbf{X}) \) are \( L_n \) Lipschitz. Indeed, we first compute the derivative of \( \hat{p}_W(\cdot; \mathbf{X}) \),
\[
\frac{d}{dx} \hat{p}_W(x; \mathbf{X}) = \frac{1}{mnh_n^2} \sum_{i \in [m], j \in [n]} K' \left( \frac{W_{i,j} + \tilde{X}_{i,j} - x}{h_n} \right), \tag{D.71}
\]
which we can see from triangle inequality that,
\[
\left\| \frac{d}{dx} \hat{p}_W(\cdot; \mathbf{X}) \right\|_\infty \leq h_n^{-2} \left\| K' \right\|_\infty \leq h_n^{-2} M, \tag{D.72}
\]
where the last inequality follows by our assumption on the kernel \( K \). This shows that the function \( \hat{p}_W(\cdot; \mathbf{X}) \) is \( L_n \) Lipschitz. The function \( \hat{q}_W(\cdot; \mathbf{X}) = \mathbb{E} \hat{p}_W(\cdot; \mathbf{X}) \) is thus also \( L_n \) Lipschitz.

Lastly, we show the desired result in Eq (D.65) by covering argument. Fix \( t > 0 \) and let \( \Delta(n, t) := L_n^{-1} (mnh_n)^{-1/2}t \). Denote \( \mathcal{D}(n, t) \) to be the minimal \( \Delta(n, t) \) cover of the interval \( I_n = [-2T_n, 2T_n] \). It is clear that, the cardinality of \( \mathcal{D}(n, t) \) satisfies,
\[
|\mathcal{D}(n, t)| \leq \Delta(n, t)^{-1} |I_n| = 4\Delta(n, t)^{-1} T_n = t^{-1} MQ_n. \tag{D.73}
\]
Now, note that, the high probability bound in Eq (D.69) holds for any fix \( x \in \mathcal{D}(n, t) \). Thus, if we denote the event \( \Lambda(n, t) \) to be,

\[
\Lambda(n, t) = \left\{ \max_{x \in \mathcal{D}(n, t)} |\hat{p}_W(x; \mathbf{X}) - \hat{q}_W(x; \mathbf{X})| \leq (mn h_n)^{-1/2} t \right\}
\]

then by union bound, we know that,

\[
P(\Lambda(n, t)^c) \leq 2|\mathcal{D}(n, t)| \exp\left(-\frac{c t^2}{M(t + M_2)}\right) \leq 2^{t^{-1}} M Q_n \exp\left(-\frac{c t^2}{M(t + M_2)}\right).
\] (D.74)

Now, the desired claim follows if we can show that, the event specified by Eq (D.65) always holds on event \( \Lambda(n, t)^c \). This is simple, as for all \( x \in I_n \), by definition there exists some \( x' \in \mathcal{D}(n, t) \) such that \(|x - x'| \leq \Delta(n, t)\). Now, the Lipschitz property of both \( \hat{p}_W(\cdot; \mathbf{X}) \) and \( \hat{q}_W(\cdot; \mathbf{X}) \) gives that,

\[
|\hat{p}_W(x; \mathbf{X}) - \hat{q}_W(x; \mathbf{X})| \leq |\hat{p}_W(x'; \mathbf{X}) - \hat{q}_W(x'; \mathbf{X})| + L_n \Delta(n, t) = 3(mnh_n)^{-1} t.
\] (D.75)

As \( x \in I_n \) is arbitrary, this gives Eq (D.65).

**Proof of Eq (D.66)** We start by evaluating the function \( \hat{q}_W(x; \mathbf{X}) \). Indeed, for \( x \in \mathbb{R} \),

\[
\hat{q}_W(x; \mathbf{X}) = \frac{1}{mn h_n} \sum_{i \in [m], j \in [n]} \int_{\mathbb{R}} K\left(\frac{w + \tilde{X}_{i,j} - x}{h_n}\right) p_W(w) \, dw
\]

\[
= \frac{1}{mn} \sum_{i \in [m], j \in [n]} \int_{\mathbb{R}} K(t)p_W(h_n t + x - \tilde{X}_{i,j}) \, dt.
\] (D.76)

Since \( K(\cdot) \) is a first order kernel, and \( \sum_{i \in [m], j \in [n]} \tilde{X}_{i,j} = 0 \), we have the identity,

\[
p_W(x) = \frac{1}{mn} \sum_{i \in [m], j \in [n]} \int_{\mathbb{R}} K(t) \left(p_W(x) - p_W'(x)(h_n t + \tilde{X}_{i,j})\right) \, dt.
\] (D.77)

Now, take difference between Eq (D.76) and Eq (D.77). Jensen’s inequality implies

\[
|\hat{q}_W(x; \mathbf{X}) - p_W(x)| \leq \frac{1}{mn} \sum_{i \in [m], j \in [n]} \int_{\mathbb{R}} K(t) \left|p_W(h_n t + x - \tilde{X}_{i,j}) - p_W(x) - p_W'(x)(h_n t + \tilde{X}_{i,j})\right| \, dt.
\] (D.78)

Now, by assumption A2, we know that, \( |p_W'(x)| \leq M_2 \) for all \( x \in \mathbb{R} \). Thus, intermediate value theorem gives that, for all \( t, x \in \mathbb{R} \) and all \( i \in [m], j \in [n] \),

\[
\left|p_W(h_n t + x - \tilde{X}_{i,j}) - p_W(x) - p_W'(x)(h_n t + \tilde{X}_{i,j})\right| \leq \frac{1}{2} M_2 (h_n t + \tilde{X}_{i,j})^2.
\] (D.79)

Plugging the estimate in Eq (D.79) into Eq (D.78), we get for \( x \in \mathbb{R} \),

\[
|\hat{q}_W(x; \mathbf{X}) - p_W(x)| \leq \frac{M_2}{2mn} \sum_{i \in [m], j \in [n]} \int_{\mathbb{R}} K(t)(h_n t + \tilde{X}_{i,j})^2 \, dt
\]

\[
= \frac{1}{2} M_2 h_n^2 \int_{\mathbb{R}} K(t)t^2 \, dt + \frac{M_2}{2mn} \sum_{i \in [m], j \in [n]} \tilde{X}_{i,j}^2.
\] (D.80)
Now, we note that, by definition of $\tilde{X}$, we know that,
\[
\sum_{i \in [m], j \in [n]} \tilde{X}_{i,j}^2 = \|\tilde{X}\|_F^2 \leq \|X\|_F^2
\] (D.81)

By our assumption on the kernel $K(\cdot)$, we see that, $\int K(t) t^2 dt \leq M$ for some $M > 0$. Thus, Eq (D.80) and Eq (D.81) together give us,
\[
\sup_{x \in \mathbb{R}} |\hat{q}_W(x; X) - p_W(x)| \leq \frac{1}{2} M_2 \left( Mh_n^2 + \frac{1}{mn} \|X\|_F^2 \right). \tag{D.82}
\]

Now, the desired claim in Eq (D.66) follows.

\section{Proof of Theorem 8}

\subsection{Proof Outline}

Throughout the proof, we denote $Z_n = (mn)^{-1/4} W_n$ (E.1)

to be the normalized noise matrix.

First, under the setting where $Z_n$ is a Gaussian random matrix, we observe that the results in [BGN12] give the desired Eq (7.27), Eq (7.28), Eq (7.29) and Eq (7.30). To be clear, an easy application of [BGN12, Theorem 2.9, 2.10] gives Eq (7.27), Eq (7.29), and Eq (7.30) when $Z_n$ is a Gaussian random matrix. Moreover, let $D : (1, \infty) \to \mathbb{R}_+$ be
\[
D(\sigma) = \frac{G^{(1)}(\sigma) \lor G^{(2)}(\sigma)}{G^{(1)}(\sigma) \land G^{(2)}(\sigma)} \tag{E.2}
\]

Then when $Z_n$ is a Gaussian random matrix, Eq (18) in the mid of the proof of [BGN12, Theorem 2.10] (see [BGN12, Section 5]) implies that for $i \in [k],$
\[
\langle \tilde{u}_i, u_i \rangle - D(\sigma_i) \langle \tilde{v}_i, v_i \rangle \overset{a.s.}{\rightarrow} 0, \tag{E.3}
\]

which together with Eq (7.29), and Eq (7.30), proves the desired Eq (7.28).

Our proof follows closely to the proof of [BGN12, Theorem 2.9, 2.10, Eq (18)]. Denote $F_{\gamma}$ to be the normalized Marchenko-Pastur distribution such that, for any subset $A \subseteq \mathbb{R},$
\[
F_{\gamma}(A) = \begin{cases} 
(1 - \gamma^{-1}) 1 \{0 \in A\} + \nu(A) & \text{if } \gamma \geq 1 \\
\nu(A) & \text{if } \gamma \leq 1,
\end{cases} \tag{E.4}
\]
where
\[ \mathrm{d}\nu(x) = \frac{1}{2\pi \gamma^{1/2}x} \sqrt{(\lambda_+ - x)(x - \lambda_-)}1 \{x \in [\lambda_-, \lambda_+]\} \]  
(E.5)

for \( \lambda_{\pm} = (\gamma^{1/4} \pm \gamma^{-1/4})^2 \). Denote \( m(z), \tilde{m}(z) : \mathbb{C}^+ \cup (-\infty, \lambda_-) \cup (\lambda_+, \infty) \to \mathbb{C} \setminus \mathbb{C}_- \) to be the Stieltjes transform of the normalized Marchenko-Pastur distribution, i.e.,
\[ m(z) = \int \frac{1}{\lambda - z} \, dF_{\gamma}(\lambda) \quad \text{and} \quad \tilde{m}(z) = \int \frac{1}{\lambda - z} \, dF_{\gamma^{-1}}(\lambda). \]  
(E.6)

Now, denote the open set \( \mathbb{C}_\gamma \) to be:
\[ \mathbb{C}_\gamma = \{ z \in \mathbb{C} : |z| > \lambda_+ = \gamma^{1/4} + \gamma^{-1/4} \}. \]  
(E.7)

A careful checking of the proof of [BGN12, Theorem 2.9, 2.10, Eq (18)] shows that it suffices to prove for any fixed sequences of unit vectors \( \{u_n\}_{n \in \mathbb{N}}, \{u'_n\}_{n \in \mathbb{N}}, \{v_n\}_{n \in \mathbb{N}} \) and \( \{v'_n\}_{n \in \mathbb{N}} \) and compact set \( S \subseteq \mathbb{C}_\gamma \), the following convergence is uniform on the set \( S \),
\[
\begin{align*}
(\text{i}) & \quad \langle u_n, (z^2 I_m - Z_n Z_n^T)^{-1} u'_n \rangle + m(z^2) \langle u_n, u'_n \rangle \xrightarrow{a.s.} 0. \\
(\text{ii}) & \quad \langle v_n, (z^2 I_n - Z_n^T Z_n)^{-1} v'_n \rangle + \tilde{m}(z^2) \langle v_n, v'_n \rangle \xrightarrow{a.s.} 0. \\
(\text{iii}) & \quad \langle u_n, (z^2 I_m - Z_n Z_n^T)^{-1} Z_n v_n \rangle \xrightarrow{a.s.} 0. \\
(\text{iv}) & \quad \langle u_n, (z^2 I_m - Z_n Z_n^T)^{-2} u'_n \rangle - m'(z^2) \langle u_n, u'_n \rangle \xrightarrow{a.s.} 0. \\
(\text{v}) & \quad \langle v_n, (z^2 I_n - Z_n^T Z_n)^{-2} Z_n^T Z_n v_n \rangle - (\tilde{m}'(z^2) z^2 + \tilde{m}(z^2)) \langle v_n, v'_n \rangle \xrightarrow{a.s.} 0. \\
(\text{vi}) & \quad \langle u_n, (z^2 I_m - Z_n Z_n^T)^{-2} Z_n v_n \rangle \xrightarrow{a.s.} 0.
\end{align*}
\]

In fact, the proof of [BGN12, Theorem 2.9, 2.10] indicates that the above coefficients in those almost sure limit (i.e., \( m(z^2) \), \( \tilde{m}(z^2) \), \( 0, m'(z^2) \), \( m'(z^2)z^2 + m(z^2) \) and \( 0 \) respectively) determine the almost sure limits of \( \{\hat{\sigma}_i\}_{i \in [r]}, \{\hat{v}_i, u_j\}_{i,j \in [r]} \) and \( \{\hat{v}_i, v_j\}_{i,j \in [r]} \). Note that, the above assertions suggest that these limit are the same as the limits that we would have if \( Z \) is Gaussian random matrix, and so consequently the almost sure limits of \( \{\hat{\sigma}_i\}_{i \in [r]}, \{\hat{v}_i, u_j\}_{i,j \in [r]} \) and \( \{\hat{v}_i, v_j\}_{i,j \in [r]} \) will be the same as if \( Z \) is Gaussian random matrix, which is given by [BGN12, Theorem 2.9, 2.10] and stated here in our desired claim Eq (7.27), Eq (7.29) and Eq (7.30).

The rest of the proofs is devoted to prove point (i) to point (vi). The organization is as follows. First, we prove point (i) to point (iii). Next, we show that point (iv) to point (vi) follows immediately from point (i) to point (iii) with the so-called derivative trick (see e.g. [DW18]).

**Proof of point (i) to point (iii)** In fact, point (i) and point (ii) follows easily from [BEK+14, Theorem 2.4, 2.5]. Thus, we only need to prove point (iii). Denote \( \tilde{Z}_n = (z^2 I_m - Z_n Z_n^T)^{-1} Z_n \). Previous results for controlling \( \langle u_n, \tilde{Z}_n v_n \rangle \) rely heavily on the concentration bounds for random \( u_n \) and \( v_n \) (See for instance [BGN12, Proposition 8.12] for details). In contrast, here \( u_n \) and \( v_n \) are fixed vectors and the only randomness here comes from the matrix \( \tilde{Z}_n \). For this reason, we adopt the moment methods to control this quantity. In execution of this moment calculations, we develop some combinatorial arguments. As the proof is a little bit lengthy, we defer it to section E.2.
Proof of point (iv) to point (vi)  Here, we employ the derivative trick [BS10b, Lemma 2.14], which is based on the Montel and Vitali’s convergence theorem from complex analysis [Rem13, Theorem 3-4, Section 7]. The idea is the following. Say we want to show point (iv), i.e., \( f_n(z) = \langle u_n, (z^2 I_m - Z_n Z_n^T)^{-2} u'_n \rangle - m'(z^2) \langle u_n, u'_n \rangle \xrightarrow{a.s.} 0 \). Then the idea is to construct a function \( F_n(z) \) such that it satisfies the following properties: (1) the derivative of \( F_n \) is precisely \( f_n \) and (2) the limit of the functions \( F_n \) is easy to evaluate. The lemma below shows that we can usually legitimately change the order between taking limits and derivatives.

Lemma E.1 (Lemma 2.14 [BS10b]). Let \( f_1, f_2, \ldots \) be analytic functions on a domain \( D \) in the complex plane satisfying \( |f_n(z)| \leq M \) for some constant \( M \) and all \( z \in D \). Suppose that there is an analytic function \( f \) on \( D \) such that \( f_n(z) \to f(z) \) for all \( z \in D \). Then it holds that \( f'_n(z) \to f'(z) \) and the convergence is uniform for any compact set \( S \subseteq D \).

Now, we show point (iv) first. To do so, consider the function

\[
F_{n,1}(t) = \langle u_n, (tI_m - Z_n Z_n^T)^{-1} u'_n \rangle + m(t) \langle u_n, u'_n \rangle. \tag{E.8}
\]

Denote the set \( \mathbb{C}_2^z = \{ z^2 : z \in \mathbb{C}_2 \} \). Fix some open set \( S \) such that its closure \( \tilde{S} \subseteq \mathbb{C}_2 \). Now that [BS10b, Theorem 5.2] shows that \( \|Z_n\|_{op} \xrightarrow{a.s.} \lambda_+ = \gamma^{1/4} + \gamma^{-1/4} \). Moreover, since \( \tilde{S} \subseteq \mathbb{C}_2^z \), we have \( \inf_{z \in \Sigma} |z| > \lambda_2^{1/2} \). Thus, for any \( \omega \in \Omega \), there exists sufficiently large \( N(= N(\omega)) \) and \( M(= M(\omega)) \) such that the function \( F_{n,1} \) is well-defined and analytic in \( S \), and moreover, for all \( n \geq N, |F_{n,1}(z)| \leq M \) for all \( z \in S \). Now that \( F_{n,1}(t) \to 0 \) for \( t \in S \) by point (i). Thus, Lemma E.1 implies that \( F_{n,1}(t) \to 0 \) for \( t \in S \), or equivalently,

\[
\langle u_n, (tI_m - Z_n Z_n^T)^{-2} u'_n \rangle - m'(t) \langle u_n, u'_n \rangle \to 0, \tag{E.9}
\]

and such convergence is uniform over \( z \in S' \) for any compact subset \( S' \subseteq S \). This gives the desired claim of point (iv).

Next, we overview the proof for point (v) and point (vi). Their proof strategies are the same as that for proving point (iv). For point (v), we start with the identity

\[
\langle v_n, (z^2 I_n - Z_n^T Z_n)^{-2} Z_n^T Z_n v'_n \rangle = z^2 \langle v_n, (z^2 I_n - Z_n^T Z_n)^{-2} v'_n \rangle - \langle v_n, (z^2 I_n - Z_n^T Z_n)^{-1} v'_n \rangle.
\]

Since point (ii) already shows that \( \langle v_n, (z^2 I_n - Z_n^T Z_n)^{-2} v'_n \rangle \to 0 \), thus, it suffices to show that,

\[
\langle v_n, (z^2 I_n - Z_n^T Z_n)^{-2} v'_n \rangle \xrightarrow{a.s.} 0. \tag{E.10}
\]

To show Eq (E.10), one needs to consider the function

\[
F_{n,2}(t) = \langle v_n, (tI_m - Z_n^T Z_n)^{-1} v'_n \rangle + \tilde{m}(t) \langle v_n, v'_n \rangle, \tag{E.11}
\]

and apply the similar derivative trick to \( F_{n,2} \), noticing that \( F_{n,2} \xrightarrow{a.s.} 0 \) by point (ii) and the class of functions \( F_{n,2} \) is uniformly bounded on any open set \( S \) such that its closure \( \tilde{S} \subseteq \mathbb{C}_2^z \). For point (vi), one instead needs to consider the function

\[
F_{n,3}(t) = \langle u_n, (tI_m - Z_n^T Z_n)^{-1} Z_n v_n \rangle + \tilde{m}(t) \langle v_n, v'_n \rangle, \tag{E.12}
\]

and apply the similar derivative trick to \( F_{n,3} \), noticing that \( F_{n,3} \xrightarrow{a.s.} 0 \) by point (iii) and the class of functions \( F_{n,3} \) is uniformly bounded on any open set \( S \) such that its closure \( \tilde{S} \subseteq \mathbb{C}_2^z \).
E.2 Proof of Point (iii)

Denote the function
\[ f_n(z) = u_n^T \left( z^2 I_{m \times m} - Z_n Z_n^T \right)^{-1} Z_n v_n. \] (E.13)

By [BS10b, Theorem 5.2], we know that \( \| Z_n \|_{op} \xrightarrow{a.s.} \lambda_+ = \gamma^{1/4} + \gamma^{-1/4} \). Hence, for any compact set \( S \subseteq \mathbb{C}_\gamma \) and \( \omega \in \Omega \), there exists some \( N(=N(\omega,S)) \) such that the function \( f_n \) is analytic in \( S \) for \( n \geq N \). Montel and Vitali’s convergence theorem for analytic functions [Rem13, Theorem 3-4, Section 7] imply that showing uniform convergence of \( f_n \) to 0 in \( S \) is equivalent to showing for any fix \( z \in S \), it holds
\[ \lim_{n \to \infty} u_n^T \left( z^2 I_{m \times m} - Z_n Z_n^T \right)^{-1} Z_n v_n \xrightarrow{a.s.} 0. \] (E.14)

As a result, our goal is to show that Eq (E.14) holds for any fix \( z \in \mathbb{C}_\gamma \).

For \( z \in \mathbb{C}_\gamma \), set \( r_z = \frac{1}{2}(|z| + \lambda_+) < |z| \) and \( \varepsilon_z = \frac{1}{2}(|z| - \lambda_+) > 0 \). Take \( N(\omega) \in \mathbb{N} \) such that \( \| Z_n \|_{op} \leq r_z \) for all \( n \geq N(\omega) \). Note that, for \( n \geq N(\omega) \), the function \( f_n(z) \) is analytic in \( \mathbb{B}(z, \varepsilon_z) \) with Laurent expansion,
\[ u_n^T \left( z^2 I_{m \times m} - Z_n Z_n^T \right)^{-1} Z_n v_n = \sum_{k=0}^{\infty} z^{-2k-2} u_n^T(Z_n Z_n^T)^k Z_n v_n. \] (E.15)

Now we prove that the desired claim of Eq (E.14) will follow if we can show that for all \( k \in \mathbb{N} \),
\[ \lim_{n \to \infty} u_n^T(Z_n Z_n^T)^k Z_n v_n \xrightarrow{a.s.} 0. \] (E.16)

In fact, denote \( A_{n,k} = z^{-2k-2} u_n^T(Z_n Z_n^T)^k Z_n v_n \). Then, for any \( \omega \in \Omega \), we have \( |A_{n,k}| \leq B_k := |z|^{-2k-2} r_z^{2k} \) for all \( n \geq N(\omega) \). Note that \( \sum_k B_k < \infty \) and for \( k \in \mathbb{N} \), \( A_{n,k} \to 0 \) by assumption. Hence, Lebesgue’s dominated convergence theorem implies that \( \lim_{n \to \infty} \sum_k A_{n,k} \to 0 \) for any \( \omega \in \Omega \), giving the desired claim in Eq (E.14) thanks to the expansion (E.15).

Now, we show for each \( k \in \mathbb{N} \), the desired Eq (E.16) holds. To simplify our notation, define the sequence \( \{c_p\}_{p \in \mathbb{N}} \) by
\[ c_p = \mathbb{E}|W_{i,j}|^p < \infty. \] (E.17)

By assumption, we know that \( c_p \) is independent of \( m, n \). As a consequence of Markov’s inequality and Borel-Cantelli Lemma, it suffices to prove that for each \( k \in \mathbb{N} \), there exists some constant \( C_k \) independent of \( n \) (but can be dependent of the sequence \( \{c_p\}_{p \in \mathbb{N}} \) such that
\[ \mathbb{E}(u_n^T(Z_n Z_n^T)^k Z_n v_n)^4 \leq C_k n^{-2}. \] (E.18)

In the rest of the proof, we show Eq (E.18). For notational simplicity, we use the compact notations \( \mathbf{u} \in \mathbb{R}^m \), \( \mathbf{v} \in \mathbb{R}^n \) and \( Z \in \mathbb{R}^{m \times n} \) to represent the vectors \( u_n, v_n \) and matrix \( Z_n \), making their dependences on \( n \in \mathbb{N} \) implicit.

The proof of Eq (E.18) is based on a moment calculation, and part of it mirrors the moment method proof for Marchenko-Pastur law [BS10b, Section 3.1]. Our first step is to expand the LHS of Eq (E.18) for fix \( k \in \mathbb{N} \). To do so, we introduce some notations. For each \( i = (i_1, \ldots, i_k) \in [m]^k \) and \( j = (j_1, \ldots, j_k) \in [n]^k \), we construct a graph \( G_1(i,j) \) in the following way. Draw two parallel lines, referring to the \( I \) line and the \( J \) line. Plot \( i_1, \ldots, i_k \) on the \( I \) line and \( j_1, \ldots, j_k \) on the \( J \) line, and draw \( k \) (down) edges from \( i_u \) to \( j_u \), \( u = 1, \ldots, k \), and \( k-1 \) (up) edges from \( j_u \) to
and $i_{u+1}, u = 1, \ldots, k$. Similarly, we denote by $G_2(j, i)$ the graph with $i_1, \ldots, i_k$ on the $I$ line, $j_1, \ldots, j_k$ on the $J$ line, $k$ (up) edges from $j_u$ to $i_u$, $u = 1, \ldots, k$, and $k - 1$ (down) edges from $i_u$ to $j_{u+1}$, $u = 1, \ldots, k - 1$. For $G_1(i, j)$ and $G_2(j, i)$, we define the scalars $Z_{G_1(i, j)}, Z_{G_2(j, i)}$ by

$$Z_{G_1(i, j)} = Z_{i_1j_1} Z_{i_2j_2} Z_{i_3j_3} \ldots Z_{i_kj_k},$$
$$Z_{G_2(j, i)} = Z_{i_1j_1} Z_{i_2j_2} Z_{i_3j_3} \ldots Z_{i_kj_k}.$$

For $\{i^{(l)}\}_{l \in [4]}$ and $\{j^{(l)}\}_{l \in [4]}$, let $G(i^{(1:4)}, j^{(1:4)})$ be the graph that is the union of the graphs $G_1(i^{(1)}, j^{(1)})$, $G_2(j^{(2)}, i^{(2)})$, $G_1(i^{(3)}, j^{(3)})$ and $G_2(j^{(4)}, i^{(4)})$. For each graph $G(i^{(1:4)}, j^{(1:4)})$, define

$$Z_{G(i^{(1:4)}, j^{(1:4)})} := Z_{G_1(i^{(1)}, j^{(1)})} \cdot Z_{G_2(j^{(2)}, i^{(2)})} \cdot Z_{G_1(i^{(3)}, j^{(3)})} \cdot Z_{G_2(j^{(4)}, i^{(4)})}.$$

![Figure 5](image-url) An example of $G(i^{(1:4)}, j^{(1:4)})$. The red, blue, orange and blue edges correspond to the edges of the graphs $G_1(i^{(1)}, j^{(1)})$, $G_2(j^{(2)}, i^{(2)})$, $G_1(i^{(3)}, j^{(3)})$ and $G_2(j^{(4)}, i^{(4)})$ respectively.

Lastly, for each $h \in [m]^4$ and $h' \in [n]^4$, we define the set of graphs

$$S(h, h') = \left\{ G(i^{(1:4)}, j^{(1:4)}) : (i^{(l)}, j^{(l)}) \in \Gamma(h, h'_l) \text{ for all } l \in [4] \right\}$$

where for $h \in [m]$ and $h' \in [n]$, the set $\Gamma(h, h')$ is defined by

$$\Gamma(h, h') = \left\{ (i, j) : i \in [m]^k, j \in [n]^k, i_1 = h, j_k = h' \right\}.$$  \hspace{1cm} (E.19)

Then we have the following expansion of the LHS of of Eq (E.18):

$$\mathbb{E} \left( u^T (ZZ^T)^k v \right)^4 = \text{tr} \left( uu^T (ZZ^T)^k Zv^T Z^T (ZZ^T)^k uu^T (ZZ^T)^k Zv^T Z^T (ZZ^T)^k \right)$$
$$= \sum_{h \in [m]^k} \sum_{h' \in [n]^k} \mathbb{E} Z_{G(i^{(1:4)}, j^{(1:4)})}.$$ \hspace{1cm} (E.20)

Now, we evaluate the RHS of Eq (E.21). For any graph $G(i^{(1:4)}, j^{(1:4)})$ that contains a single edge, i.e., an edge not coincident with another edge, then we shall have

$$\mathbb{E} Z_{G(i^{(1:4)}, j^{(1:4)})} = 0$$ \hspace{1cm} (E.22)

since the random variables $Z_{i,j}$ are assumed to be independent and mean 0. Now, for any $h \in [m]^4$ and $h' \in [n]^4$, denote

$$S^h(h, h') = S(h, h') \cap \{ G(i^{(1:4)}, j^{(1:4)}) : G(i^{(1:4)}, j^{(1:4)}) \text{ does not have a single edge} \}.$$
by Eq (E.22), we have
\[
\sum_{G(i^{(1:4)}j^{(1:4)}) \in S(h,h')} \mathbb{E} Z_{G(i^{(1:4)}j^{(1:4)})} = \sum_{G(i^{(1:4)}j^{(1:4)}) \in S(h,h')} \mathbb{E} Z_{G(i^{(1:4)}j^{(1:4)})}. \tag{E.23}
\]

By Hölder’s inequality, we know that, each \(Z_{G(i^{(1:4)}j^{(1:4)})}\) satisfies
\[
|\mathbb{E} Z_{G(i^{(1:4)}j^{(1:4)})}| \leq \mathbb{E} \left[ (|Z_{i,j}|^{2k+1})^4 \right] \leq c_{4(2k+1)}(mm)^{-(2k+1)}. \tag{E.24}
\]

Hence, if we set \(N^s(h,h') = |S^s(h,h')|\), then Eq (E.23) and Eq (E.24) show that
\[
\sum_{G(i^{(1:4)}j^{(1:4)}) \in S(h,h')} \mathbb{E} Z_{G(i^{(1:4)}j^{(1:4)})} \leq C_k n^{-4k-2} N^s(h,h'). \tag{E.25}
\]
for some constant \(C_k > 0\). Plugging it into Eq (E.21), we get that
\[
\mathbb{E} \left( u^T (ZZ)^k Z v \right)^4 \leq C_k n^{-4k-2} \sum_{h \in [m]^4} N^s(h,h') \prod_{l \in [4]} |u_{h_l} v_{h'_l}|. \tag{E.26}
\]

Now, for \(p \in \{m, n\}\), decompose the set \([p]^4 = \cup_{l \in [5]} T_{l,p}\), where we define \(\{T_{l,p}\}_{l \in [5]}\) by

1. \(T_{1,p} = \{l \in [p]^4 : |l| = 4\}\)
2. \(T_{2,p} = \{l \in [p]^4 : |l| = 3\}\)
3. \(T_{3,p} = \{l \in [p]^4 : |l| = 2, \text{some entry of } (l_1, l_2, l_3, l_4) \text{ has multiplicity 3}\}\)
4. \(T_{4,p} = \{l \in [p]^4 : |l| = 2, \text{each entry of } (l_1, l_2, l_3, l_4) \text{ has multiplicity 2}\}\)
5. \(T_{5,p} = \{l \in [p]^4 : |l| = 1\}\).

Then by Eq (E.26), we have that
\[
\mathbb{E} \left( u^T (ZZ)^k Z v \right)^4 \leq C_k n^{-4k-2} \sum_{l_1 \in [5]} M_m(l_1) M_n(l_2) N^s(l_1, l_2), \tag{E.27}
\]
where for each \(l_1, l_2 \in [5]\), we define
\[
M_m(l_1) := \sum_{h \in \mathcal{T}_{l_1, m}} \prod_{l \in [4]} |u_{h_l}|, \quad M_n(l_2) := \sum_{h \in \mathcal{T}_{l_2, n}} \prod_{l \in [4]} |v_{h_l}| \quad \text{and} \quad N^s(l_1, l_2) := \sup_{h \in \mathcal{T}_{l_1, m}, h' \in \mathcal{T}_{l_2, n}} N^s(h, h'). \tag{E.28}
\]

Now, motivated by Eq (E.27), we provide bounds on \(M_p(l)\) for \(p \in \{m, n\}\) and \(l \in [5]\) and on \(N(l_1, l_2)\) for \(l_1, l_2 \in [5]\).
Bounds on $M_p(l)$ We have the following result.

**Lemma E.2.** For some universal constant $C$, the bounds below hold for $p \in \{m, n\}$,

$$M_p(l) \leq C \cdot \begin{cases} p^2 & \text{if } l = 1 \\ p & \text{if } l = 2 \\ p^{1/2} & \text{if } l = 3 \\ 1 & \text{if } l \in \{4, 5\} \end{cases}. \quad (E.29)$$

**Proof** We only prove the case where $p = m$. The crux of the proof is the following: for any $l \in \mathbb{N}$

$$\sum_i |u_i|^l \leq \begin{cases} 1 & \text{if } l \geq 2 \\ m^{1/2} & \text{if } l = 1. \end{cases} \quad (E.30)$$

In fact, when $l \geq 2$, since the vector $u$ lies on the unit sphere, we know that $\sum_i |u_i|^l \leq \sum_i |u_i|^2 \leq 1$. On the other hand, when $l = 1$, then we have $\sum_i |u_i| \leq m^{1/2} \sum_i |u_i|^2 \leq m^{1/2}$ by Cauchy-Schwartz inequality. Now, using Eq (E.30), it is easy to enumerate all the possibilities of $M_m(l)$:

1. $M_m(1) \leq (\sum_i |u_i|)^4 \leq m^2$.
2. $M_m(2) \leq \left(\frac{4}{3}\right) (\sum_i |u_i|)^2 (\sum_i |u_i|^2) \leq 6m$.
3. $M_m(3) \leq \left(\frac{4}{3}\right) (\sum_i |u_i|) (\sum_i |u_i|^3) \leq 4m^{1/2}$.
4. $M_m(4) \leq \left(\frac{4}{3}\right) (\sum_i |u_i|^2)^2 \leq 6$.
5. $M_m(5) \leq \sum_i |u_i|^4 \leq 1$.

This gives the desired claim of the lemma. \hfill \Box

Bounds on $N^s(l_1, l_2)$ We start by bounding $N^s(h, h')$ for $h \in [m]^4, h' \in [n]^4$. Define for $l \in \mathbb{Z}^4$

$$|l| = |\{l_1, l_2, l_3, l_4\}|. \quad (E.31)$$

To bound $N^s(h, h')$, we find the following characteristics useful:

- Denote by $\chi_{\text{nav}}(h, h')$ the maximum number of vertices of any graph in the set $S^s(h, h')$.
- Denote by $\chi_{\text{edge}}(h, h')$ the maximum number of edges of any graph in the set $S^s(h, h')$.
- Denote by $\chi_{\text{cc}}(h, h')$ the maximum number of components of any graph in the set $S^s(h, h')$.
- Define $\chi_{\text{nv}}(h, h') = |h| + |h'|$.

Our next lemma provides an upper bound onto $N^s(h, h')$ based on the above characteristics.

**Lemma E.3.** There exists some constant $C_k > 0$ such that

$$N^s(h, h') \leq C_k n(\chi_{\text{nav}} - \chi_{\text{nv}})(h, h') \leq C_k n(\chi_{\text{edge}} + \chi_{\text{cc}} - \chi_{\text{nv}})(h, h'). \quad (E.32)$$

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For any two graphs $G, G'$, call $l$ by permuting lab($G$) to lab($G'$) without permuting the indices $h$ and $h'$. Denote $C_{eqv}(h, h')$ to be the number of different isomorphism class and $N_{eqv}(h, h')$ to be the maximum of size of any isomorphism class. Now, we show the following three claims:

1. For some constant $C_k$ depending solely on $k$, $C_{eqv}(h, h') \leq C_k$.
2. $N_{eqv}(h, h') \leq (m + n)(\chi_{nav} - \chi_{nav})(h, h')$ and $(\chi_{edge} + \chi_{cc}) \leq 16k$.
3. $\chi_{nav}(h, h') \leq (\chi_{edge} + \chi_{cc})(h, h')$.

Clearly, the desired claim of the lemma follows by the above two claims since we have by definition

$$N^g(h, h') \leq C_{eqv}(h, h') N_{eqv}(h, h').$$

Now, we prove the above claims. In fact, since $|\text{lab}(G)| \leq 8k$, the number of $C_{eqv}(h, h')$ should be only dependent of $k$ and independent of $n$. This gives claim 1. To show the first part of claim 2, we note that the size of the isomorphism class is bounded by the number of possible labeling that does not change the labels $h$ and $h'$, which can be easily shown upper bounded by $(m + n)(\chi_{nav} - \chi_{nav})(h, h')$. This proves the first part of claim 2. The second part of claim 2 follows by definition of the set $S^g(h, h')$. Finally, since for any graph $G$, its number of vertices is upper bounded by the sum of its number of edges and connected components. This gives the third claim 3.

Following Lemma E.3, our next lemma gives generic upper bounds onto $\chi_{edge}(h, h')$ and $\chi_{cc}(h, h')$. Call $1 \in \mathbb{Z}^4$ odd, if say for some $j \in [4]$, $\{i \in [4] : l_i = l_j\}$ is odd.

**Lemma E.4.** For $h \in [m]^4$ and $h' \in [n]^4$, we have

$$\chi_{edge}(h, h') \leq 4k + 2 \quad \text{and} \quad \chi_{cc}(h, h') \leq \min\{|h|, |h'|\}. \quad (E.33)$$

In addition,

1. if either $h$ or $h'$ is odd, then

$$\chi_{edge}(h, h') \leq 4k + 1. \quad (E.34)$$

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2. if either $|h| = 4$ or $|h'| = 4$, then
\[
\chi_{\text{edge}}(h, h') \leq 4k. \quad (E.35)
\]

**Proof** The first part of Eq (E.33) follows from the fact that any graph in the set $S^s(h, h')$ has in total $8k + 4$ edges with each edge appearing at least twice. The second part of Eq (E.33) follows by the construction of graph in the set $S^s(h, h')$. To prove Eq (E.34), note that, when $h$ is odd, say $h_l (l \in [4])$ appears odd times in $(h_1, h_2, h_3, h_4)$, it means that some edge connecting $h_l$ must appear at least 3 times. Since each edge at least appears twice and there are in total $8k + 4$ edges, we get that the number of total edges in this case is upper bounded by $4k + 1$, giving Eq (E.34). To prove Eq (E.35), we note that, when $|h| = 4$, then for each $l \in [4]$, there exists some edge connecting $h_l$ that appears at least three times. This shows that at least four different edges appear at least 3 times. Since each edge at least appears twice and there are in total $8k + 4$ edges, we get that the number of total edges in this case is upper bounded by $4k$, giving Eq (E.35).

Lastly, we provide an estimate on $\chi_{\text{nav}}(h, h')$ when $h \in T_{(5,m)}$ and $h' \in T_{(5,n)}$.

**Lemma E.5.** For $h \in T_{(5,m)}$, $h' \in T_{(5,n)}$, we have
\[
\chi_{\text{nav}}(h, h') \leq 4k + 2. \quad (E.36)
\]

**Proof** In fact, any graph $G \in S^s(h, h')$ has at most $4k + 2$ distinct edges. Based on this, we divide our discussion into two cases.

1. The graph $G$ has exactly $4k + 2$ distinct edges. Hence, each edge has multiplicity 2. Now consider the undirected graph $\tilde{G}$ induced by $G$: it consists of the vertices and the $4k + 2$ different edges of $G$. We claim that there exists a loop in $\tilde{G}$. This implies the existence of loop in the graph $\tilde{G}$.

![Figure 7](image_url). An illustration of $G$ and $\tilde{G}$ when the graph $G$ has exactly $4k + 2$ distinct edges. The left plot is the directed graph $G$ and the right plot is the induced undirected graph $\tilde{G}$. Note that $\tilde{G}$ contains a loop.

To show that $\tilde{G}$ contains a loop, let $h = (h, h, h, h) \in [m]^4$ and $h' = (h', h', h', h') \in [n]^4$ for some $h \in [m]$ and $h' \in [n]$. In fact, note that, by construction, in graph $G$, the out-degree of vertex $h$ is at least 2 and the in-degree of vertex $h'$ is at least 2. Thus, there are two different path connecting $h$ and $h'$ in $\tilde{G}$. This implies the existence of loop in the graph $\tilde{G}$.
2. The graph $G$ has no more than $4k+1$ different edges. In this case, since the graph is connected, the number of different vertices is at most $4k + 2$.

This concludes the proof.

Now, we summarize Lemma E.3, Lemma E.4 and Lemma E.5 to upper bound $N^s(l_1, l_2)$.

**Lemma E.6.** For some constant $C_k$, the bounds below hold for all $l_1, l_2 \in [4]$:

$$N^s(l_1, l_2) \leq C_k n \chi(l_1, l_2),$$

where

$$\chi(l_1, l_2) = \begin{cases} 
4k - 4 & \text{if } \min\{l_1, l_2\} = 1 \\
4k - 2 & \text{if } \min\{l_1, l_2\} = 2 \\
4k - 1 & \text{if } \min\{l_1, l_2\} = 3 \\
4k & \text{if } \min\{l_1, l_2\} = 4 \\
4k & \text{if } \min\{l_1, l_2\} = 5.
\end{cases}$$

**Proof** Define the following quantities

$$\chi_{\text{nav}}(l_1, l_2) = \sup_{h \in T(l_1, m), h' \in T(l_2, n)} \chi_{\text{nav}}(h, h'),$$

$$\chi_{\text{edge}}(l_1, l_2) = \sup_{h \in T(l_1, m), h' \in T(l_2, n)} \chi_{\text{edge}}(h, h'),$$

$$\chi_{\text{cc}}(l_1, l_2) = \sup_{h \in T(l_1, m), h' \in T(l_2, n)} \chi_{\text{cc}}(h, h'),$$

$$\chi_{\text{nv}}(l_1, l_2) = \inf_{h \in T(l_1, m), h' \in T(l_2, n)} \chi_{\text{nv}}(h, h').$$

By Lemma E.3, we know that, for all $l_1, l_2 \in [5]$,

$$N^s(l_1, l_2) \leq C_k n \chi_{\text{edge}}(l_1, l_2) + \chi_{\text{cc}}(l_1, l_2) - \chi_{\text{nv}}(l_1, l_2).$$

By Lemma E.4, we can easily upper bound $\chi_{\text{edge}}(l_1, l_2), \chi_{\text{cc}}(l_1, l_2), \chi_{\text{nv}}(l_1, l_2)$, and we enumerate those upper bound in the table below (note by symmetry between $l_1$ and $l_2$, we only list the bounds
for \( l_1 \leq l_2 \) \[
\begin{array}{cccc}
(l_1, l_2) & \chi_{\text{edge}}(l_1, l_2) & \chi_{\text{cc}}(l_1, l_2) & \chi_{\text{nv}}(l_1, l_2) \\
(1, 1) & \leq 4k & \leq 4 & 8 \\
(1, 2) & \leq 4k & \leq 3 & 7 \\
(2, 2) & \leq 4k + 1 & \leq 3 & 6 \\
(1, 3) & \leq 4k & \leq 2 & 6 \\
(2, 3) & \leq 4k + 1 & \leq 2 & 5 \\
(3, 3) & \leq 4k + 1 & \leq 2 & 4 \\
(1, 4) & \leq 4k & \leq 2 & 6 \\
(2, 4) & \leq 4k + 1 & \leq 2 & 5 \\
(3, 4) & \leq 4k + 1 & \leq 2 & 4 \\
(4, 4) & \leq 4k + 2 & \leq 2 & 4 \\
(1, 5) & \leq 4k & 1 & 5 \\
(2, 5) & \leq 4k + 1 & 1 & 4 \\
(3, 5) & \leq 4k + 1 & 1 & 3 \\
(4, 5) & \leq 4k + 2 & 1 & 3 \\
(5, 5) & \leq 4k + 2 & 1 & 2 \\
\end{array}
\] (E.44)

Now, plugging the above estimates into Eq (E.43), it is easy to check that, we have for \( l_1, l_2 \in [5] \),
\[
N^s(l_1, l_2) \leq C_k n \chi'(l_1, l_2). \tag{E.45}
\]

where
\[
\chi'(l_1, l_2) = \begin{cases} 
\chi(l_1, l_2) & \text{if } \min\{l_1, l_2\} \leq 4 \\
4k + 1 & \text{if } \min\{l_1, l_2\} = 5.
\end{cases} \tag{E.46}
\]

Note that, we can improve the estimate in Eq (E.45) when \( l_1 = l_2 = 5 \). In fact, by Lemma E.5, we know that \( \chi_{\text{nv}}(5, 5) \leq 4k + 2 \). Hence, by Lemma E.3 we get that,
\[
N^s(5, 5) \leq C_k n \chi_{\text{nv}}(5, 5) - \chi_{\text{nv}}(5, 5) \leq C_k n^{4k}. \tag{E.47}
\]

Now, the desired claim of the lemma follows by Eq (E.45) and Eq (E.47). \( \square \)

**Summary** Now, back to Eq (E.27). By Lemma E.2 and Lemma E.6, we can check easily that, there exists some constant \( C_k > 0 \) such that for all \( l_1, l_2 \in [5] \),
\[
M_m(l_1) M_n(l_2) N^s(l_1, l_2) \leq C_k n^{-2}. \tag{E.48}
\]

Substituting the above bound into Eq (E.27) concludes the proof of Point (iii).

### F Proof of Theorem 5

By Theorem 7, \( \hat{X}^{(0)}(Y) = \hat{f}_{Y,\varepsilon}(Y) \) has the decomposition below:
\[
\hat{X}^{(0)}(Y) = I_W X + \sqrt{I_W} Z + \Delta, \tag{F.1}
\]
where $\Delta$ is a random matrix satisfying
\[
\lim_{\varepsilon \to 0} \lim_{n \to \infty, m/n \to \gamma} \frac{1}{(mn)^{1/4}} \|\Delta\|_{\text{op}} = 0
\] (F.2)

and $Z$ is some random matrix, whose entries are i.i.d bounded with mean 0 variance 1 and moreover, for some constants $\varepsilon_0, C > 0$ independent of $m, n$, we have $\|Z\|_{\text{max}} \leq C\varepsilon^{-1}$ for all $\varepsilon \leq \varepsilon_0$. For notational simplicity, in the rest of the proof, denote $\tilde{X}^{(0)}(Y) \in \mathbb{R}^{m \times n}$ and its SVD decomposition
\[
\tilde{X}^{(0)}(Y) = X + \frac{1}{\sqrt{I_W}} Z \quad \text{and} \quad \tilde{X}^{(0)}(Y) = (mn)^{1/4} \tilde{U} \tilde{\Sigma}^{(0)} \tilde{V}^T.
\] (F.3)

Let $\tilde{U}_l = (\tilde{u}_1, \ldots, \tilde{u}_l)^T \in \mathbb{R}^{m \times l}$ be the matrix consisting of the top $l$ left singular vectors of $\tilde{X}^{(0)}(Y)$.

We divide our proof into two cases. In the first case, we assume additionally that the top singular values $\{\sigma_i\}_{i \in [l]}$ are pairwise different, i.e.,
\[
\sigma_1 > \sigma_2 > \ldots > \sigma_l.
\] (F.4)

Fix $\varepsilon > 0$. By Theorem 8, we can without loss of generality (by flipping the sign of $\{\tilde{u}_i\}$ if necessary) assume that, for any $i,j \in [l]$,
\[
\lim_{n \to \infty} \tilde{u}_i^T \tilde{u}_j \overset{a.s.}{\to} \begin{cases} G(\sigma_i; I_W) & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}
\] (F.5)

As a consequence, we have,
\[
\left\| \tilde{U}_l^T U_l - \text{diag}(G(\sigma_i; I_W))_{i \in [l]} \right\|_{\text{op}} \overset{a.s.}{\to} 0,
\]
which, by Weyl’s inequality, implies that,
\[
\sigma_{\min}(\tilde{U}_l^T U_l) \overset{a.s.}{\to} G(\sigma_i; I_W).
\] (F.6)

To pass the result in Eq (F.6) to $\sigma_{\min}(\tilde{U}_l^T U_l)$, our idea is to view $\tilde{X}^{(0)}(Y)$ as a perturbed version of $\tilde{X}^{(0)}(Y)$ and then do some perturbation analysis to show that
\[
\sigma_{\min}(\tilde{U}_l^T U_l) \approx \sigma_{\min}(\tilde{U}_l^T U_l).
\] (F.7)

More precisely, first, we note that
\[
\sigma_{\min}(\tilde{U}_l^T U_l)^2 = \sigma_{\min}(U_l^T \tilde{U}_l \tilde{U}_l^T U_l) \quad \text{and} \quad \sigma_{\min}(\tilde{U}_l^T U_l)^2 = \sigma_{\min}(U_l^T \tilde{U}_l \tilde{U}_l^T U_l).
\]

Then using Weyl’s inequality and noting that $U_l$ is unitary, we get that
\[
\sigma_{\min}(\tilde{U}_l^T U_l)^2 \in \left[ \sigma_{\min}(\tilde{U}_l^T U_l)^2 \pm \left\| U_l \tilde{U}_l^T - \tilde{U}_l \tilde{U}_l^T \right\|_{\text{op}} \right],
\] (F.8)
Now, by assumption $\sigma_l > \sigma_{l+1}$ and $\sigma_l > I_{W}^{1/2}$. Thus Theorem 8 (or Lemma A.4) implies for some constant $\vartheta_0 > 0$ (independent of $\varepsilon, \delta$),

$$\lim_{n \to \infty, m/n \to \gamma} \frac{1}{(mn)^{1/4}} \left( \sigma_l(\tilde{X}^{(0)}(Y)) - \sigma_{l+1}(\tilde{X}^{(0)}(Y)) \right) \geq \vartheta_0. \quad (F.9)$$

Fix this $\vartheta_0$. Next, by Eq (F.2), we know for any $\tilde{\Delta} > 0$, there exists some $\varepsilon_0$ such that for $\varepsilon \leq \varepsilon_0$

$$\lim_{n \to \infty, m/n \to \gamma} \frac{1}{(mn)^{1/4}} \|\Delta\|_{op} \leq \tilde{\Delta}. \quad (F.10)$$

According to Eq (F.9) and Eq (F.10), we apply the Davis-Kahan Theorem (see lemma I.5) to see that for any $\tilde{\Delta}$ satisfying $\tilde{\Delta} < \vartheta_0/2$, there exists some $\varepsilon_0$ such that for all $\varepsilon < \varepsilon_0$,

$$\lim_{n \to \infty, m/n \to \gamma} \|\tilde{U}_l U_l^T - \tilde{U}_l \tilde{U}_l^T\|_{op} \leq \frac{\Delta}{\vartheta_0 - 2\Delta}. \quad (F.11)$$

Since $\tilde{\Delta} < \vartheta_0/2$ is arbitrary, we take $\tilde{\Delta} \to 0$ on both sides and get that

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty, m/n \to \gamma} \|\tilde{U}_l U_l^T - \tilde{U}_l \tilde{U}_l^T\|_{op} = 0. \quad (F.12)$$

By Eq (F.6), Eq (F.8) and Eq (F.12), we see that,

$$\sigma_{\min}\left(\tilde{U}_l^T U_l\right) \xrightarrow{a.s.} G(\sigma_l; I_{W}), \quad (F.13)$$

giving the desired result of the Theorem.

In the second case, we consider the situation where some elements of $\{\sigma_i\}_{i \in [k]}$ coincide. The idea is to reduce the proof of the second case to the first case, through the perturbation trick that we shall describe. Indeed, for any $\{\iota_i\}_{i \in [l]}$ such that $\{\sigma_i + \iota_i\}_{i \in [l]}$ are distinct, we define

$$X(\iota) = X + (mn)^{1/4} \sum_{i=1}^r \iota_i u_i u_i^T. \quad (F.14)$$

Now, for such $\{\iota_i\}_{i \in [l]}$, denote $\tilde{X}(Y; \iota)$ and its SVD decomposition

$$\tilde{X}^{(0)}(Y; \iota) = I_{W} X(\iota) + \sqrt{I_{W}} Z + \Delta \quad \text{and} \quad \tilde{X}^{(0)}(Y; \iota) = (mn)^{1/4} \tilde{U}(\iota) \Sigma(\iota) \tilde{V}(\iota)^T. \quad (F.15)$$

Denote analogously $\tilde{U}_l(\iota)$ to be the matrix consisting of the top $l$ singular vectors of $\tilde{X}^{(0)}(Y; \iota)$ and $\sigma_i(\iota)$ to be the top $i$th singular value of $X(\iota)$ for each $i$. Let $\iota^{\max} = \max_{i \in [l]} |\iota_i|$. Since by assumption $\sigma_l > \sigma_{l+1}$, Weyl’s Theorem implies when $\iota^{\max}$ is small enough, then the set of the top $l$ singular values of $X(\iota)$ is precisely the set $\{\sigma_i + \iota_i\}_{i \in [l]}$. Moreover, the top $l$ singular values of $X(\iota)$ are pairwise different by our choice of $\{\iota_i\}_{i \in [l]}$. Thus, we may use the established result in the first case to conclude that,

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty, m/n \to \gamma} \sigma_{\min}\left(\tilde{U}_l(\iota^T) U_l\right) = G(\sigma_l(\iota); I_{W}). \quad (F.16)$$
Now that, by a similar argument proving Eq (F.8) we can show that,

$$\sigma_{\min} \left( \tilde{U}_l^T (\iota) U_l \right)^2 \in \left[ \sigma_{\min} \left( U_l^T U_l \right)^2 \pm \| \tilde{U}_l (\iota) \tilde{U}_l^T (\iota) - \tilde{U}_l \tilde{U}_l^T \|_{\text{op}} \right],$$  \hspace{1cm} (F.17)

Moreover, Eq (F.2), Eq (F.9) and Weyl’s inequality, we have, for some constant $\varepsilon_0, \vartheta_0 > 0$ (independent of $m, n$), we have for all $\varepsilon \leq \varepsilon_0$,

$$\lim_{n \to \infty, m/n \to \gamma} \frac{1}{(mn)^{1/4}} \left( \sigma_l (\hat{X}^{(0)} (Y)) - \sigma_{l+1} (\hat{X}^{(0)} (Y)) \right) \geq \delta_0'. \hspace{1cm} (F.18)$$

Now, viewing $\hat{X}^{(0)} (Y; \iota)$ as a perturbed version of $\hat{X}^{(0)} (Y)$, Eq (F.18) and Davis-Kahan-Theorem imply that for all $\iota$ such that $\iota^{\text{max}} < \vartheta_0$, we have for $\varepsilon \leq \varepsilon_0$,

$$\lim_{n \to \infty, m/n \to \gamma} \left\| \tilde{U}_l (\iota) \tilde{U}_l^T (\iota) - \tilde{U}_l \tilde{U}_l^T \right\|_{\text{op}} \leq \frac{\iota^{\text{max}}}{\vartheta_0 - 2 \iota^{\text{max}}}. \hspace{1cm} (F.19)$$

By letting $\varepsilon \to 0$ and $\iota^{\text{max}} \to 0$, we get

$$\lim_{\iota \to 0} \lim_{\varepsilon \to 0} \lim_{n \to \infty, m/n \to \gamma} \left\| \tilde{U}_l (\iota) \tilde{U}_l^T (\iota) - \tilde{U}_l \tilde{U}_l^T \right\|_{\text{op}} = 0. \hspace{1cm} (F.20)$$

By Eq (F.16), Eq (F.17) and Eq (F.20), we know that,

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty, m/n \to \gamma} \sigma_{\min} \left( \tilde{U}_l^T U_l \right) = \lim_{\iota \to 0} \lim_{\varepsilon \to 0} \lim_{n \to \infty, m/n \to \gamma} \sigma_{\min} \left( \tilde{U}_l (\iota) U_l \right)$$

$$= \lim_{\iota \to 0} G (\sigma_l (\iota); I_W) = G (\sigma_l; I_W) \hspace{1cm} (F.21)$$

where the last identity since the function $\sigma \to G (\sigma; I_W)$ is continuous on $[l_W^{-1/2}, \infty)$. This concludes the proof of Theorem 5.

G Proof of Theorem A.1

Define $\{p_1, p_2, \ldots, p_k\}$ and $\{q_1, q_2, \ldots, q_k\}$ by the columns of $P_k$, $Q_k$ respectively, i.e.,

$$P_k = [p_1, p_2, \ldots, p_k] \text{ and } Q_k = [q_1, q_2, \ldots, q_k]. \hspace{1cm} (G.1)$$

For each $l < k$, we define the matrices $P_l \in \mathbb{R}^{m \times l}$ and $Q_l \in \mathbb{R}^{n \times l}$ by

$$P_l = [p_1, p_2, \ldots, p_l] \text{ and } Q_l = [q_1, q_2, \ldots, q_l]. \hspace{1cm} (G.2)$$

Similarly, we define the vectors $\{\tilde{p}_l\}_{l \in [k]}$, $\{\tilde{q}_l\}_{l \in [k]}$ and the matrices $\tilde{P}_l \in \mathbb{R}^{m \times l}$, $\tilde{Q}_l \in \mathbb{R}^{n \times l}$. For each $l \leq k$, we define the error matrices $\Delta_l \in \mathbb{R}^{m \times n}$ and singular gap $\delta_l \in \mathbb{R}_+$ by

$$\Delta_l = P_l Q_l^T - \tilde{P}_l \tilde{Q}_l^T, \text{ and } \delta_l = \sigma_l (A) - \sigma_{l+1} (A) \hspace{1cm} (G.3)$$

By assumption in Eq (A.30), we know that $\delta_k > \vartheta > 2 \| E \|_{\text{op}}$. 

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As our starting point, we prove the following claim. For each \( l \in [k] \) such that \( \delta_l > 2 \|E\|_{\text{op}} \),

\[
\|\Delta_l\|_{\text{op}} \leq \frac{2}{\delta_l} \|E\|_{\text{op}}. \tag{G.4}
\]

To do so, define the matrices \( A^{\text{sym}}, \tilde{A}^{\text{sym}}, E^{\text{sym}} \in \mathbb{R}^{(m+n) \times (m+n)} \) by

\[
A^{\text{sym}} := \begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix}, \quad \tilde{A}^{\text{sym}} := \begin{bmatrix} 0 & \tilde{A} \\ \tilde{A}^T & 0 \end{bmatrix} \quad \text{and} \quad E^{\text{sym}} := \begin{bmatrix} 0 & E \\ E^T & 0 \end{bmatrix}. \tag{G.5}
\]

Now that since \( \tilde{A} = A + E \), we know that \( \tilde{A}^{\text{sym}} = A^{\text{sym}} + E^{\text{sym}} \). By standard result in matrix analysis [SS90, Theorem 4.2], we know that the top \( k+1 \) eigenvalues of \( A^{\text{sym}} \) are precisely the top \( k+1 \) singular values of \( A \), and moreover, for any \( l \in [k] \), the top \( l \) eigenvectors of \( A^{\text{sym}} \) and \( \tilde{A}^{\text{sym}} \) are columns of \( P^{\text{sym}}, \tilde{P}^{\text{sym}} \in \mathbb{R}^{(m+n) \times k} \) defined below

\[
P_l^{\text{sym}} := \begin{bmatrix} P_l \\ Q_l \end{bmatrix} \quad \text{and} \quad \tilde{P}_l^{\text{sym}} := \begin{bmatrix} \tilde{P}_l \\ \tilde{Q}_l \end{bmatrix}. \tag{G.6}
\]

Now, we fix any \( l \) such that \( \delta_l > 2 \|E\|_{\text{op}} \). By Davis-Kahan Theorem (see Lemma I.5), we have

\[
\|\tilde{P}_l^{\text{sym}} (\tilde{P}_l^{\text{sym}})^T - P_l^{\text{sym}} (P_l^{\text{sym}})^T\|_{\text{op}} \leq \frac{\|E^{\text{sym}}\|_{\text{op}}}{\delta_l - \|E^{\text{sym}}\|_{\text{op}}}, \tag{G.7}
\]

which is equivalent to

\[
\left\| \begin{bmatrix} P_l P_l^T - \tilde{P}_l \tilde{P}_l^T \\ Q_l Q_l^T - \tilde{Q}_l \tilde{Q}_l^T \end{bmatrix} \right\|_{\text{op}} \leq \frac{\|E\|_{\text{op}}}{\delta_l - \|E\|_{\text{op}}}. \tag{G.8}
\]

Now that the bound above implies

\[
\|\Delta_l\|_{\text{op}} \leq \frac{\|E\|_{\text{op}}}{\delta_l - \|E\|_{\text{op}}}. \tag{G.9}
\]

Since \( \delta_l > 2 \|E\|_{\text{op}} \), Eq (G.9) implies the desired claim at Eq (G.4).

Now, we are ready to show the desired claim of Theorem A.1. Define the auxiliary matrix

\[
B_k = \tilde{P}_k f(S_k) \tilde{Q}_k. \tag{G.10}
\]

By triangle inequality, we have,

\[
\|f(A_k) - f(\tilde{A}_k)\|_{\text{op}} \leq \|E_1\|_{\text{op}} + \|E_2\|_{\text{op}} \quad \text{where} \quad E_1 = f(\tilde{A}_k) - B_k \quad \text{and} \quad E_2 = f(A_k) - B_k. \tag{G.11}
\]

We bound error matrix \( E_1 \) first. By definition, we have

\[
\|E_1\|_{\text{op}} = \|f(\tilde{A}_k) - B_k\|_{\text{op}} = \|\tilde{P}_k (f(S_k) - f(\tilde{S}_k)) \tilde{Q}_k\|_{\text{op}} \leq \|f(S_k) - f(\tilde{S}_k)\|_{\text{op}}. \tag{G.12}
\]
Note that $\|S_k - \tilde{S}_k\|_{\text{op}} \leq \|E\|_{\text{op}}$ by Weyl’s inequality. Thus, the assumption that $f$ is $(L, \alpha)$ Hölder continuous on $[\tau, \zeta]$ for $\zeta > \sigma_1(A)$ implies that
\begin{equation}
\|f(S_k) - f(\tilde{S}_k)\|_{\text{op}} \leq L \|S_k - \tilde{S}_k\|_{\text{op}} \leq L \|E\|_{\text{op}}. \tag{G.13}
\end{equation}
Substituting the above estimate into Eq (G.12) gives the upper bound
\begin{equation}
\|E_1\|_{\text{op}} \leq L \|E\|_{\text{op}}^\alpha. \tag{G.14}
\end{equation}
Next, we bound the error matrix $E_2$. Indeed, by definition, we have,
\begin{equation*}
f(A_k) - B_k = \sum_{l \in [k]} f(\sigma_l(A)) \left( p_l q_l^T - \tilde{p}_l \tilde{q}_l^T \right)
= \sum_{l < k} (f(\sigma_l(A)) - f(\sigma_{l+1}(A))) \left( p_l Q_l^T - \tilde{p}_l \tilde{Q}_l^T \right) + f(\sigma_k(A)) \left( p_k Q_k^T - \tilde{p}_k \tilde{Q}_k^T \right).
\end{equation*}
Hence, by triangle inequality, we get the estimate below,
\begin{equation}
\|E_2\|_{\text{op}} \leq \sum_{l < k} \left| f(\sigma_l(A)) - f(\sigma_{l+1}(A)) \right| \left\| P_l Q_l^T - \tilde{P}_l \tilde{Q}_l^T \right\|_{\text{op}} + \left\| f(\sigma_k(A)) \right\| \left\| P_k Q_k^T - \tilde{P}_k \tilde{Q}_k^T \right\|_{\text{op}}. \tag{G.15}
\end{equation}
Now that $\delta_k = \sigma_k(A) - \sigma_{k+1}(A) > \vartheta > 2 \|E\|_{\text{op}}$ by assumption. Eq (G.4) shows that,
\begin{equation}
\left\| P_k Q_k^T - \tilde{P}_k \tilde{Q}_k^T \right\|_{\text{op}} \leq \frac{2}{\delta_k} \|E\|_{\text{op}} \leq \frac{2}{\vartheta} \|E\|_{\text{op}}. \tag{G.16}
\end{equation}
Now, we show that for any $l < k$,
\begin{equation}
\left| f(\sigma_l(A)) - f(\sigma_{l+1}(A)) \right| \left\| P_l Q_l^T - \tilde{P}_l \tilde{Q}_l^T \right\|_{\text{op}} \leq 4L \|E\|_{\text{op}}^\alpha. \tag{G.17}
\end{equation}
To show Eq (G.17), we divide it into two cases.

1. In the first case, we assume that $\delta_l = \sigma_l(A) - \sigma_{l+1}(A) \leq 2 \|E\|_{\text{op}}$. Since $f$ is $(L, \alpha)$ Hölder continuous (see Eq (A.29)) for $\alpha \in (0, 1]$, we know that,
\begin{equation}
\left| f(\sigma_l(A)) - f(\sigma_{l+1}(A)) \right| \leq L \left( 2 \|E\|_{\text{op}} \right)^\alpha \leq 2L \|E\|_{\text{op}}^\alpha. \tag{G.18}
\end{equation}
Since $P_l, \tilde{P}_l, Q_l$ and $\tilde{Q}_l$ are all orthonormal, we know that,
\begin{equation}
\left\| P_l Q_l^T - \tilde{P}_l \tilde{Q}_l^T \right\|_{\text{op}} \leq \left\| P_l Q_l^T \right\|_{\text{op}} + \left\| \tilde{P}_l \tilde{Q}_l^T \right\|_{\text{op}} \leq 2. \tag{G.19}
\end{equation}
Now the desired claim at Eq (G.17) follows by Eq (G.18) and Eq (G.19).

2. In the second case, we assume that $\delta_l = \sigma_l(A) - \sigma_{l+1}(A) > 2 \|E\|_{\text{op}}$. Since $f$ is $(L, \alpha)$ Hölder continuous (see Eq (A.29)), we know that,
\begin{equation}
\left| f(\sigma_l(A)) - f(\sigma_{l+1}(A)) \right| \leq L \left( \sigma_l(A) - \sigma_{l+1}(A) \right)^\alpha \leq L \delta_l^\alpha. \tag{G.20}
\end{equation}
Since $\delta_l > 2\|E\|_{op}$ and $\alpha \in (0, 1]$, Eq (G.4) shows that,

$$\left\| P_lQ_l^T - \tilde{P}_l\tilde{Q}_l^T \right\|_{op} \leq \frac{2}{\delta_l} \|E\|_{op} \leq \frac{2}{\delta_l} \|E\|_{op}. \quad (G.21)$$

Now the desired claim at Eq (G.17) follows by Eq (G.20) and Eq (G.21).

Substituting the bound at Eq (G.17) and Eq (G.16) into Eq (G.15), we get that,

$$\|E_2\|_{op} \leq 4(k - 1)L \|E\|_{op} + \frac{2}{\delta_l} \|f(\sigma_k(A))\|E\|_{op}. \quad (G.22)$$

Now the desired claim of Theorem A.1 follows by plugging Eq (G.14) and Eq (G.22) into Eq (G.11).

**H Proofs of Technical Lemma**

**H.1 Proof of Lemma C.1**

We start by proving the case where $X$ is symmetric (in this case $m = n$). In this case, we define matrices $\{X^{(i,j)}\}_{1 \leq i \leq j \leq n}$ such that,

$$X^{(i,j)}_{k,l} = \begin{cases} X_{i,j} & \text{if } (k,l) = (i,j) \text{ or } (k,l) = (j,i) \\ 0 & \text{otherwise} \end{cases}.$$ 

Note that, $\{X^{(i,j)}\}_{1 \leq i \leq j \leq n}$ are all symmetric matrices, mean 0, independent to each other and

$$X = \sum_{1 \leq i \leq j \leq n} X^{(i,j)}.$$ 

Denote $\{\varepsilon_{i,j}\}_{1 \leq i \leq j \leq n}$ be independent Radamacher random variables and $\tilde{X}^{(i,j)} = \varepsilon_{i,j}X^{(i,j)}$. By standard symmetrization argument, we have

$$\mathbb{E} \|X\|_{op}^2 = \mathbb{E} \left\{ \left\| \sum_{1 \leq i \leq j \leq n} X^{(i,j)} \right\|_{op}^2 \right\} \leq 4\mathbb{E} \left\{ \left\| \sum_{1 \leq i \leq j \leq n} \tilde{X}^{(i,j)} \right\|_{op}^2 \right\}. \quad (H.1)$$

Now, $\{\tilde{X}^{(i,j)}\}_{1 \leq i \leq j \leq n}$ are symmetrically distributed symmetric matrices. We may use Lemma I.4 to get for some numerical constant $C > 0$,

$$\mathbb{E} \left\{ \left\| \sum_{1 \leq i \leq j \leq n} \tilde{X}^{(i,j)} \right\|_{op}^k \right\} \leq C^k \left[ (\log n + k)^{1/2} \times \mathbb{E} \left\{ \sum_{1 \leq i \leq j \leq n} \tilde{X}^{(i,j)}^2 \right\}_{op} + (\log n + k) \times \mathbb{E} \left\{ \max_{1 \leq i \leq j \leq n} \|\tilde{X}^{(i,j)}\|_{op} \right\} \right]^k$$

$$\leq C^k k^k \log^k (n) \left( \max_{i \in [n]} \mathbb{E} \left[ \sum_{j=1}^n X_{i,j}^2 \right] + \mathbb{E} \left[ \max_{1 \leq i \leq j \leq n} |X_{i,j}| \right] \right)^k \quad (H.2)$$
This proves the result for symmetric matrix $X \in \mathbb{R}^{n \times n}$. The more general situation where $X \in \mathbb{R}^{m \times n}$ is asymmetric can be reduced to the symmetric case. In fact, define $X^{\text{sym}} \in \mathbb{R}^{(m+n) \times (m+n)}$ as

$$X^{\text{sym}} := \begin{bmatrix} 0 & X^\top \\ X & 0 \end{bmatrix}.$$  

(H.3)

Then $X^{\text{sym}}$ is symmetric and satisfies $\|X^{\text{sym}}\|_{\text{op}} = \|X\|_{\text{op}}$. Now, by applying the already established result to the symmetric matrix $X^{\text{sym}} \in \mathbb{R}^{(m+n) \times (m+n)}$, we get the desired claim of the lemma for the asymmetric matrix $X$.

### H.2 Proof of Lemma D.4

First, we have,

$$g(s_1, t_1) - g(s_2, t_2) = \frac{s_2 - s_1}{t_1 + \varepsilon} + \frac{s_2(t_1 - t_2)}{(t_1 + \varepsilon)(t_2 + \varepsilon)}.$$  

Since $t_1, t_2 \geq 0$, by triangle inequality, we get,

$$|g(s_1, t_1) - g(s_2, t_2)| \leq \varepsilon^{-1} |s_1 - s_2| + \varepsilon^{-2}|s_2||t_1 - t_2|.$$  

Similarly,

$$|g(s_1, t_1) - g(s_2, t_2)| \leq \varepsilon^{-1} |s_1 - s_2| + \varepsilon^{-2}|s_1||t_1 - t_2|.$$  

The last two inequalities together give the desired claim.

### H.3 Proof of Lemma D.5

By Assumption A2, we know that for all $i \in [m], j \in [n],$

$$|g^2(p_W(W_{i,j}), p_W(W_{i,j}))| \leq \varepsilon^{-2} \|p_W(\cdot)\|^2_\infty \leq \varepsilon^{-2}M_2^2.$$  

(H.4)

Thus, since $\bar{I}_{W,\varepsilon}$ is the average of the random variables $g^2(p_W(W_{i,j}), p_W(W_{i,j}))$, we know by Hoeffding’s inequality [BLM13, Theorem 2.8] that, for all $t > 0,$

$$P \left( |\bar{I}_{W,\varepsilon} - \mathbb{E}\bar{I}_{W,\varepsilon}| > t \right) \leq 2 \exp \left( -\frac{mnt^2\varepsilon^4}{2M_2^4} \right).$$  

(H.5)

Now, we show the below crucial estimate,

$$|\mathbb{E}\bar{I}_{W,\varepsilon} - I_{W,\varepsilon}| \leq \delta_{W,\varepsilon}.$$  

(H.6)

Indeed, Eq (H.6) follows by the direction computations below

$$|\mathbb{E}\bar{I}_{W,\varepsilon} - I_{W,\varepsilon}| = \left| \int_{\mathbb{R}} \frac{(p_W(w))^2}{(p_W(w) + \varepsilon)^2p_W(w)dw} - \int_{\mathbb{R}} \frac{(p_W(w))^2}{p_W(w) + \varepsilon}dw \right| = \varepsilon \int_{\mathbb{R}} \frac{(p_W(w))^2}{p_W(w) + \varepsilon}dw = \delta_{W,\varepsilon}.$$  

(H.6)

Hence, Eq (H.5) and Eq (H.6) together show that, for all $t > 0,$

$$P \left( |\bar{I}_{W,\varepsilon} - I_{W,\varepsilon}| > t + \delta_{W,\varepsilon} \right) \leq P \left( |\bar{I}_{W,\varepsilon} - \mathbb{E}\bar{I}_{W,\varepsilon}| > t \right) \leq 2 \exp \left( -\frac{mnt^2\varepsilon^4}{2M_2^4} \right),$$  

giving the desired claim of the lemma.
H.4 Proof of Lemma D.6

To simplify the notations, we introduce the quantities \( \{G_{1,i,j}\}_{i \in [m], j \in [n]} \) to be,

\[
G_{1,i,j} := \left| g(\hat{p}_Y(Y_{i,j}), \hat{p}_Y(Y_{i,j})) - g(p_W(W_{i,j}), p_W(W_{i,j})) \right|
\]

and the quantities \( \{G_{2,i,j}\}_{i \in [m], j \in [n]} \) to be,

\[
G_{2,i,j} := \left| g^2(\hat{p}_Y(Y_{i,j}), \hat{p}_Y(Y_{i,j})) - g^2(p_W(W_{i,j}), p_W(W_{i,j})) \right|.
\]

Moreover, we denote \( G_1^2 \) to be the mean of \( \{G_{2,i,j}^2\}_{i \in [m], j \in [n]} \), i.e,

\[
G_1^2 = \frac{1}{mn} \sum_{i \in [m], j \in [n]} G_{2,i,j}^2. \tag{H.8}
\]

As our starting point, we see by triangle inequality that,

\[
\mathcal{E}_2 = \frac{1}{mn} \sum_{i \in [m], j \in [n]} \left| g^2(\hat{p}_Y(Y_{i,j}), \hat{p}_Y(Y_{i,j})) - g^2(p_W(W_{i,j}), p_W(W_{i,j})) \right| \leq \frac{1}{mn} \sum_{i \in [m], j \in [n]} G_{2,i,j}. \tag{H.9}
\]

We next upper bound the RHS of Eq (H.9). Note the elementary inequality below,

\[ |s^2 - t^2| \leq (s - t)^2 + 2|s - t|(|s| + |t|) \]

for \( s, t \in \mathbb{R} \).

If we apply it to \( s = g(\hat{p}_Y(Y_{i,j}), \hat{p}_Y(Y_{i,j})) \) and \( t = g(p_W(W_{i,j}), p_W(W_{i,j})) \), we get that,

\[
G_{2,i,j} \leq 2 \left| g(p_W(W_{i,j}), p_W(W_{i,j})) \right| G_{1,i,j} + G_{2,i,j}^2, \tag{H.10}
\]

Therefore, if we plug the individual estimate of Eq (H.10) into RHS of Eq (H.9), we get that,

\[
\mathcal{E}_2 \leq \frac{2}{mn} \sum_{i,j} \left| g(p_W(W_{i,j}), p_W(W_{i,j})) \right| G_{1,i,j} + \frac{1}{mn} \sum_{i,j} G_{2,i,j} \leq 2 \tilde{T}_{W,\varepsilon}^{1/2} \times G_1 + G_1^2, \tag{H.11}
\]

where the last inequality follows by Cauchy-Schwartz inequality. Now, to show the desired claim of the lemma, it suffices to show for some constant \( C > 0 \) the bound below on \( G_1 \):

\[
G_1 \leq C \left( \varepsilon^{-2} T_{n,1} + \varepsilon^{-1} T_{n,2} \right). \tag{H.12}
\]

Indeed, to show Eq (H.12), we first upper each term \( G_{1,i,j} \). By Lemma D.4,

\[
G_{1,i,j} \leq \varepsilon^{-1} \left| \hat{p}_Y(Y_{i,j}) - p_W(W_{i,j}) \right| + \varepsilon^{-2} \| p_W(\cdot) \|_{\infty} \left| \hat{p}_Y(Y_{i,j}) - p_W(W_{i,j}) \right| \tag{H.13}
\]

Note that \( \| p_W(\cdot) \|_{\infty} \leq M_2 \) by Assumption A2. Hence, by Jensen’s inequality,

\[
G_{1,i,j}^2 \leq C \left[ \varepsilon^{-2} \left( \hat{p}_Y(Y_{i,j}) - p_W(W_{i,j}) \right)^2 + \varepsilon^{-4} \left( \hat{p}_Y(Y_{i,j}) - p_W(W_{i,j}) \right)^2 \right]
\]

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for some constant $C > 0$. Summing over all $i \in [m], j \in [n]$, we get that,

$$G_1^2 \leq C \left[ \frac{1}{\varepsilon^2 mn} \sum_{i \in [m], j \in [n]} \left( \hat{p}_Y(\tilde{Y}_{i,j}) - p_W(W_{i,j}) \right)^2 + \frac{1}{\varepsilon^4 mn} \sum_{i \in [m], j \in [n]} \left( \hat{p}_Y(\tilde{Y}_{i,j}) - p_W(W_{i,j}) \right)^2 \right]. \quad (H.14)$$

Now, we bound each of the two individual summation terms in the above parentheses. In fact, we show that, for some constant $C > 0$,

$$\frac{1}{mn} \sum_{i \in [m], j \in [n]} \left( \hat{p}_Y(\tilde{Y}_{i,j}) - p_W(W_{i,j}) \right)^2 \leq C \left( \left\| \hat{p}_Y(\tilde{Y}) - p_W(\tilde{Y}) \right\|_{\text{max}}^2 + \frac{1}{mn} \left\| X \right\|_F^2 + \left\| W \right\|_F^2 \right), \quad (H.15)$$

and

$$\frac{1}{mn} \sum_{i \in [m], j \in [n]} \left( \hat{p}_Y(\tilde{Y}_{i,j}) - p_W(W_{i,j}) \right)^2 \leq C \left( \left\| \hat{p}_Y(\tilde{Y}) - p_W(\tilde{Y}) \right\|_{\text{max}}^2 + \frac{1}{mn} \left\| X \right\|_F^2 + \left\| W \right\|_F^2 \right). \quad (H.16)$$

The proof of Eq (H.15) and Eq (H.16) is similar, and we exemplify the proof by showing Eq (H.15). Indeed, by triangle inequality, we have, for $i \in [m], j \in [n]$,

$$\left| \hat{p}_Y(\tilde{Y}_{i,j}) - p_W(W_{i,j}) \right| \leq \left| \hat{p}_Y(\tilde{Y}_{i,j}) - p_W(\tilde{Y}_{i,j}) \right| + \left| p_W(\tilde{Y}_{i,j}) - p_W(W_{i,j}) \right|.$$

Now, since $\left\| p_W(\cdot) \right\|_{\infty} \leq M_2$ by Assumption A2, we have,

$$\left| \hat{p}_Y(\tilde{Y}_{i,j}) - p_W(W_{i,j}) \right| \leq \left\| \hat{p}_Y(\tilde{Y}) - p_W(\tilde{Y}) \right\|_{\text{max}} + M_2 \left| \tilde{Y}_{i,j} - W_{i,j} \right|.$$

Now, we sum up the above inequality for $i \in [m], j \in [n]$. Jensen’s inequality implies,

$$\frac{1}{mn} \sum_{i \in [m], j \in [n]} \left( \hat{p}_Y(\tilde{Y}_{i,j}) - p_W(W_{i,j}) \right)^2 \leq 2 \left( \left\| \hat{p}_Y(\tilde{Y}) - p_W(\tilde{Y}) \right\|_{\text{max}}^2 + \frac{M_2^2}{mn} \left\| \tilde{Y} - W \right\|_F^2 \right).$$

The desired claim of Eq (H.15) now follows by the above estimate and

$$\frac{1}{mn} \left\| \tilde{Y} - W \right\|_F^2 = \frac{1}{mn} \left\| \tilde{X} - W 1_m 1_n^T \right\|_F^2 \leq 2 \left( \left\| W \right\|^2 + \frac{1}{mn} \left\| X \right\|_F^2 \right) \leq 2 \left( \left\| W \right\|^2 + \frac{1}{mn} \left\| X \right\|_F^2 \right). \quad (H.17)$$

Now, the bound of $G_1$ in Eq (H.12) follows by plugging Eq (H.15) and Eq (H.16) into Eq (H.14) (and note that the RHS of both Eq (H.15) and Eq (H.16) are simply $C T_{n,1}^2$ and $C T_{n,2}^2$).

**H.5 Proof of Lemma D.7**

By definition, $\bar{W}$ is the average of $mn$ independent mean $0$ random variables $\{W_{i,j}\}_{i \in [m], j \in [n]}$. By Assumption A1, we know that $\mathbb{E}[W_{i,j}]^2 \leq M_1$, and thus

$$\mathbb{E}W^2 \leq (mn)^{-1} M_1.$$

Hence, Markov’s inequality implies for all $t > 0$

$$P \left( |\bar{W}| \geq t(mn)^{-1/2} \right) \leq t^{-2} M_1. \quad (H.18)$$
H.6 Proof of Lemma D.8

Note that, Assumption A1 gets $E|W_{i,j}|^2 \leq M_1$. Thus, for all $t > 0$,

$$P(\|W\|_{\max} \geq t) \leq \sum_{i \in [m], j \in [n]} P(|W_{i,j}| \geq t) \leq t^{-2} \sum_{i \in [m], j \in [n]} E|W_{i,j}|^2 \leq t^{-2}mnM_1. \quad (H.19)$$

H.7 Proof of Lemma D.9

The proof is a simple application of Bernstein’s inequality (see Lemma I.2). Introduce the centered random variables below for $i \in [n]$ (for notational simplicity):

$$\tilde{K}_{h,X_i}(x) := K_{h,X_i}(x) - EK_{h,X_i}(x).$$

By triangle inequality, the random variables $\tilde{K}_{h,X_i}(x)$ are bounded random variables as

$$|\tilde{K}_{h,X_i}(x)| \leq |K_{h,X_i}(x)| + |EK_{h,X_i}(x)| \leq 2 \|K(\cdot)\|_{\infty}.$$

In addition, we have the following upper bound on the second moment of $|\tilde{K}_{h,X_i}(x)|$

$$E|\tilde{K}_{h,X_i}(x)|^2 \leq EK_{h,X_i}(x) = \int_R K^2\left(\frac{x-w}{h}\right)p_{X_i}(w) \, dw = h \int_R K^2(z)p_{X_i}(x-hz) \, dz \leq h\sigma^2 \|p_{X_i}(\cdot)\|_{\infty},$$

where the first inequality follows by the fact that $\tilde{K}_{h,X_i}(x)$ centers $K_{h,X_i}(x)$ and the second inequality follows by the definition of $\sigma^2$ and $\|p_{X_i}(\cdot)\|_{\infty}$. Now, we apply Bernstein’s inequality (i.e, Lemma I.2) to the independent random variables $\tilde{K}_{h,X_i}(x)$ and get that, for all $t > 0$,

$$P(|Z_n(x) - EZ_n(x)| \geq t) \leq 2 \exp\left(-\frac{nht^2}{\sigma^2p_{\infty} + M_{\infty}t}\right).$$

The desired claim of the lemma now thus follows.

I Some useful tools

Lemma I.1 follows from Marcinkiewicz-Zygmund inequality and Rosenthal’s inequality (see [BLM13, Theorem 15.11]).

**Lemma I.1.** Let $Z_1, Z_2, \ldots, Z_n$ be independent random variables such that $EZ_i = 0$ for $i \in [n]$. Then, for any fix $q \geq 2$, there exists some constant $C_q > 0$ depending only on $q$ such that,

$$\mathbb{E} \left| \sum_{i=1}^{n} Z_i \right|^q \leq C_q \left[ \left( \mathbb{E} \left( \sum_{i=1}^{n} Z_i^2 \right)^{q/2} \right) + \sum_{i=1}^{n} \mathbb{E} |Z_i|^q \right]$$

The next lemma is a restatement of Bernstein’s inequality.
**Lemma I.2** (Bernstein’s inequality). Let \( Z_1, Z_2, \ldots, Z_n \) be \( n \) independent random variables with mean \( \mathbb{E}[Z_i] = 0 \) for all \( i \in [n] \). Suppose that, almost surely \( |Z_i| \leq M \) for some (non-random) \( M > 0 \). Then, for some numerical constant \( c > 0 \), the following inequality holds for all \( t > 0 \),

\[
\mathbb{P} \left( \frac{1}{n} \sum_{i=1}^{n} Z_i \geq t \right) \leq 2 \exp \left( -\frac{c t^2}{n \mathbb{E}[Z_i^2] + Mt} \right).
\]

The next Lemma I.3 is a standard result in matrix analysis [SS90, Thm 2.11].

**Lemma I.3.** For any matrix \( A \in \mathbb{R}^{n \times m} \), we have,

\[
\|A\|_{\text{op}} \leq \|A\|_{\ell_1 \rightarrow \ell_1} \|A\|_{\ell_\infty \rightarrow \ell_\infty} \leq \sqrt{mn} \|A\|_{\text{max}}.
\]

The next Lemma I.4 gives a moment bound on the operator norm of independent sums of random matrices. See [CGT12, Theorem A.1(2)] for a proof.

**Lemma I.4.** Let \( X_i \in \mathbb{R}^{d \times d} \) be independent and symmetrically distributed Hermitian matrices. Then, for \( k \geq 2 \) and \( r \geq \max\{k, 2 \log d\} \),

\[
\mathbb{E} \left[ \left\| \frac{1}{n} \sum_{i=1}^{n} X_i \right\|_{\text{op}} \right]^{1/k} \leq \sqrt{2er} \left( \left\| \sum_{i=1}^{n} \mathbb{E}[X_i^2] \right\|_{\text{op}} \right)^{1/2} + 2er \left( \mathbb{E} \max_{i \in [n]} \|X_i\|_{\text{op}}^k \right)^{1/k}. \quad (I.1)
\]

Recall the following version of Davis-Kahan sin \( \Theta \) theorem [SS90, Theorem 4.4]

**Lemma I.5** (Davis-Kahan). Let \( \hat{A} = A + E \in \mathbb{R}^{p \times q} \). For \( s \leq \min(p, q) \), let \( U \) and \( \hat{U} \) be the first \( s \) columns of the left singular matrices of \( A \) and \( \hat{A} \) respectively. Denote by \( \gamma = \sigma_s(A) - \sigma_{s+1}(A) \). If \( \gamma \geq 2 \|E\|_{\text{op}} \), we have the following inequality

\[
\sqrt{1 - \sigma_{\text{min}}^2(\hat{U}^\top U)} \leq \frac{\|E\|_{\text{op}}}{\gamma - \|E\|_{\text{op}}}.
\]

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