Supplement to
Permutation Testing for Dependence in Time Series

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S.1 Introduction

In this supplement, we provide proofs of the results introduced in the main paper, in addition to statements and proofs of auxiliary results used in the proofs of the lemmas and theorems in the main paper.

S.2 Permutation distribution for m-dependent sequences

Proof of Theorem 2.1. By Theorem 1.1 in Rinott (1994), itself from Stein (1986) page 110, for $Y_i = X_i/\sigma_n$,

$$\begin{aligned} |\mathbb{P}(W_n \le t) - \Phi(t)| \le 2\mathbb{E}\left[\left(\sum_{i=1}^n \sum_{j \in S_i} (Y_i Y_j - \mathbb{E} Y_i Y_j)\right)^2\right]^{1/2} + \\ + \sqrt{\frac{\pi}{2}} \mathbb{E} \sum_{i=1}^n |\mathbb{E}[Y_i \,|\, Y_j : j \notin S_i]| + \\ + 2^{3/4} \pi^{-1/4} \mathbb{E}\left[\sum_{i=1}^n |Y_i| \left(\sum_{j \in S_i} Y_j\right)^2\right]^{1/2}. \end{aligned} \tag{S.1}$$

We now bound the expectations on the right hand side of (S.1). Begin by bounding the second expectation. By definition of the S_i,

$$\begin{aligned}\mathbb{E}\sum_{i=1}^{n}\left|\mathbb{E}\left[Y_i \,|\, Y_j : j \notin S_i\right]\right| &= \mathbb{E}\sum_{i=1}^{n}\left|\mathbb{E}Y_i\right| \\ &= 0 \ ,\end{aligned} \tag{S.2}$$

since $\mathbb{E}X_i = 0$ for all i.

We turn our attention to the third expectation. Let $i \in [n]$.

$$\begin{aligned}\mathbb{E}\left[\left(\sum_{j\in S_i} Y_j\right)^4\right] &\leq \mathbb{E}\left[\left(\sum_{j\in S_i} |Y_j|\right)^4\right] \\ &\leq D^4\mathbb{E}\left[\max_{j\in S_i} Y_j^4\right] \quad (x \mapsto x^4 \text{ is increasing on } \mathbb{R}_{\geq 0}) \\ &\leq D^4\mathbb{E}\left[\sum_{j\in S_i} Y_j^4\right] \quad (\text{union bound}) \\ &\leq \frac{D^5 M_4}{\sigma_n^4} \ .\end{aligned} \tag{S.3}$$

Hence

$$\begin{aligned}\mathbb{E}\left[\sum_{i=1}^{n}|Y_i|\left(\sum_{j\in S_i} Y_j\right)^2\right] &= \sum_{i=1}^{n}\mathbb{E}\left[|Y_i|\left(\sum_{j\in S_i} Y_j\right)^2\right] \\ &\leq \sum_{i=1}^{n}\mathbb{E}\left[Y_i^2\right]^{1/2}\mathbb{E}\left[\left(\sum_{j\in S_i} Y_j\right)^4\right]^{1/2} \quad (\text{by Cauchy-Schwarz}) \\ &\leq \sum_{i=1}^{n}\left(\frac{M_2}{\sigma_n^2}\cdot\frac{D^5 M_4}{\sigma_n^4}\right)^{1/2} \quad (\text{by (S.3)}) \\ &= \frac{n\left(D^5 M_2 M_4\right)^{1/2}}{\sigma_n^3} \ .\end{aligned} \tag{S.4}$$

Lastly, we bound the first term in the expectation. Let $U_{ij} = Y_iY_j - \mathbb{E}Y_iY_j$, $i, j \in [n]$.

$$\begin{aligned}\mathbb{E}\left[\left(\sum_{i=1}^{n}\sum_{j\in S_i}(Y_iY_j - \mathbb{E}Y_iY_j)\right)^2\right] &= \mathbb{E}\left[\sum_{i=1}^{n}\sum_{j\in S_i}U_{ij}\sum_{k=1}^{n}\sum_{l\in S_k}U_{kl}\right] \\ &= \sum_{i=1}^{n}\sum_{j\in S_i}\sum_{k=1}^{n}\sum_{l\in S_k}\mathbb{E}U_{ij}U_{kl} \ .\end{aligned} \tag{S.5}$$

By independence, and the fact that $\mathbb{E}U_{ij} = 0$, observe that $\mathbb{E}U_{ij}U_{kl} = 0$ unless either k or l is a member of $S_i \cup S_j$. Note that $|S_i \cup S_j| \leq 2D$. It follows that we have at most $4nD^3$ nonzero terms on the right hand side of (S.5). This is because there are n choices for i, at most D choices for j, at most $2D$ choices for k, under the assumption that $k \in S_i \cup S_j$, and at most D choices for l, and the same upper bound on nonzero terms if we assume that $l \in S_i \cup S_j$ (noting that $l \in S_k \iff k \in S_l$). Also, by repeated application of Cauchy-Schwarz,

$$\begin{aligned}
|\mathbb{E}\left[(Y_iY_j - \mathbb{E}Y_iY_j)(Y_kY_l - \mathbb{E}Y_kY_l)\right]| &\leq \mathbb{E}\left[(Y_iY_j - \mathbb{E}Y_iY_j)^2\right]^{1/2} \mathbb{E}\left[(Y_kY_l - \mathbb{E}Y_kY_l)^2\right]^{1/2} \\
&\leq \mathbb{E}\left[Y_i^2Y_j^2\right]^{1/2} \mathbb{E}\left[Y_k^2Y_l^2\right]^{1/2} \\
&\leq \mathbb{E}\left[Y_i^4\right]^{1/4} \mathbb{E}\left[Y_j^4\right]^{1/4} \mathbb{E}\left[Y_k^4\right]^{1/4} \mathbb{E}\left[Y_l^4\right]^{1/4} \\
&\leq \frac{M_4}{\sigma_n^4} \, .
\end{aligned} \tag{S.6}$$

Substituting the result of (S.6) into (S.5), we obtain that

$$\mathbb{E}\left[\left(\sum_{i=1}^n \sum_{j \in S_i} (Y_iY_j - \mathbb{E}Y_iY_j)\right)^2\right] \leq \frac{4nD^3M_4}{\sigma_n^4} \, . \tag{S.7}$$

Combining (S.2), (S.4), and (S.7), and substituting into (S.1):

$$\begin{aligned}
|\mathbb{P}\left(W_n \leq t\right) - \Phi(t)| &\leq 2\left(\frac{4nD^3M_4}{\sigma_n^4}\right)^{1/2} + 2^{3/4}\pi^{-1/4}\left(\frac{n\left(D^5M_2M_4\right)^{1/2}}{\sigma_n^3}\right)^{1/2} \\
&= \frac{1}{\sigma_n}\left(4\left(\frac{nD^3M_4}{\sigma_n^2}\right)^{1/2} + 2^{3/4}\pi^{-1/4}\left(\frac{n}{\sigma_n}\right)^{1/2}\left(D^5M_2M_4\right)^{1/4}\right) \, ,
\end{aligned}$$

as required. ■

Proof of Theorem 2.2, in the i.i.d. setting. Let $\Pi_n, \Pi_n' \overset{i.i.d.}{\sim} \text{Unif}\{S_n\}$, independent of $X_1, \ldots, X_n$. Let $\Pi_n\mathbf{X}$, $\Pi_n'\mathbf{X}$ denote the action of Π_n and Π_n' on $\mathbf{X} = (X_1, \ldots, X_n)$, respectively. By Lehmann and Romano (2005) Theorem 15.2.3, it suffices to show that Hoeffding's condition holds, i.e. that $\sqrt{n}\hat{\rho}\left(\Pi_n\mathbf{X}\right)$ and $\sqrt{n}\hat{\rho}\left(\Pi_n'\mathbf{X}\right)$ are asymptotically independent, with common distribution Φ.

Note that, for all $m \in \mathbb{R}$,

$$\left(\sqrt{n}\hat{\rho}\left(\Pi_n\mathbf{X}\right), \sqrt{n}\hat{\rho}\left(\Pi_n'\mathbf{X}\right)\right) \overset{d}{=} \left(\sqrt{n}\hat{\rho}\left(\Pi_n(\mathbf{X}+m)\right), \sqrt{n}\hat{\rho}\left(\Pi_n'(\mathbf{X}+m)\right)\right) ,$$

and so, taking $m = -\mu$, we may assume, without loss of generality, that $\mathbb{E}X_i = 0$. Similarly, since $\hat{\rho}$ is invariant under the transformation $\mathbf{X} \mapsto \tau\mathbf{X}$ for all $\tau > 0$, we may also assume, without loss of generality, that $\sigma^2 = 1$.

Note also that, for all $\pi_n \in S_n$,

$$\bar{X}_{\pi_n} := \frac{1}{n}\sum_{i=1}^{n} X_{\pi_n^{-1}(i)} = \bar{X}_n$$
$$\hat{\sigma}^2_{\pi_n} := \frac{1}{n}\sum_{i=1}^{n}\left(X_{\pi_n^{-1}(i)} - \bar{X}_{\pi_n}\right)^2 = \hat{\sigma}^2_n \ .$$

Since $\hat{\sigma}^2_n \to 1$ almost surely by the strong law of large numbers (under the assumption of finite fourth moments), it follows, by Slutsky's theorem for randomization distributions (see Theorem 5.2 in Chung and Romano (2013)), that $(\sqrt{n}\hat{\rho}\,(\Pi_n\mathbf{X})\,,\,\sqrt{n}\hat{\rho}\,(\Pi'_n\mathbf{X})) \xrightarrow{d} N(0,\, I_2)$ if and only if

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n-1}\begin{pmatrix}(X_{\Pi_n^{-1}(i)} - \bar{X}_n)(X_{\Pi_n^{-1}(i+1)} - \bar{X}_n)\\ (X_{\Pi_n'^{-1}(i)} - \bar{X}_n)(X_{\Pi_n'^{-1}(i+1)} - \bar{X}_n)\end{pmatrix} \xrightarrow{d} N(0,\, I_2) \ . \tag{S.8}$$

Also, by the moment assumptions, $\bar{X}_n = O_p(1/\sqrt{n})$ by the Central Limit Theorem, and $\sum_{i=1}^{n-1} X_{\Pi_n^{-1}(i)} = n\bar{X}_n + O_p(1)$. Hence

$$\begin{aligned}\frac{1}{\sqrt{n}}\sum_{i=1}^{n-1}(X_{\Pi_n^{-1}(i)} - \bar{X}_n)(X_{\Pi_n^{-1}(i+1)} - \bar{X}_n) &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n-1} X_{\Pi_n^{-1}(i)}X_{\Pi_n^{-1}(i+1)} - \sqrt{n}\bar{X}_n^2 + o_p(1)\\ &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n-1} X_{\Pi_n^{-1}(i)}X_{\Pi_n^{-1}(i+1)} + o_p(1) \ ,\end{aligned}$$

and similarly for $\Pi'_n\mathbf{X}$. Hence the convergence in (S.8) holds if and only if

$$\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n-1} X_{\Pi_n^{-1}(i)}X_{\Pi_n^{-1}(i+1)},\ \frac{1}{\sqrt{n}}\sum_{i=1}^{n-1} X_{\Pi_n'^{-1}(i)}X_{\Pi_n'^{-1}(i+1)}\right) \xrightarrow{d} N(0,\, I_2) \ . \tag{S.9}$$

Since the X_i are i.i.d.,we observe that $\Pi_n\mathbf{X}$ and $\mathbf{X}$ are identically distributed. Hence it is easy to see, by relabelling, that $(\Pi_n\mathbf{X},\, \Pi'_n\mathbf{X}) \stackrel{d}{=} (\mathbf{X},\, \Pi'_n\Pi_n^{-1}\mathbf{X})$. Also, $\Pi_n^{-1} \stackrel{d}{=} \Pi_n \stackrel{d}{=} \Pi'_n \stackrel{d}{=} \Pi'_n\Pi_n^{-1}$, and so the convergence in (S.9) holds if and only if

$$\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n-1} X_iX_{i+1},\ \frac{1}{\sqrt{n}}\sum_{i=1}^{n-1} X_{\Pi_n(i)}X_{\Pi_n(i+1)}\right) \xrightarrow{d} N(0,\, I_2) \ . \tag{S.10}$$

The sequence $\{X_iX_{i+1},\, i \in [n-1]\}$ has uniformly bounded fourth moment, by the moment condition and independence of the X_i. Observe that the sequence $\{X_iX_{i+1},\, i \in [n-1]\}$ is 1-dependent, with mean 0, variance 1, finite 4th moment, and first-order autocorrelation 0. Therefore, by the m-dependent central limit theorem and Slutsky's theorem (rescaling from $\sqrt{n-1}$ to $\sqrt{n}$), as $n \to \infty$,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n-1} X_i X_{i+1} \xrightarrow{d} N(0,1) \ . \tag{S.11}$$

A similar result holds for the corresponding sum of the sequence $\Pi_n \mathbf{X}$. By the Cramér-Wold device, (S.10) holds if and only if, for all a, $b \in \mathbb{R}$,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n-1} a X_i X_{i+1} + \frac{1}{\sqrt{n}} \sum_{i=1}^{n-1} b X_{\Pi_n(i)} X_{\Pi_n(i+1)} \xrightarrow{d} N(0,\, a^2 + b^2) \ . \tag{S.12}$$

Let $Y_i = \frac{1}{\sqrt{n}} a X_i X_{i+1} + \frac{1}{\sqrt{n}} b X_{\Pi_n(i)} X_{\Pi_n(i+1)}$, $i \in [n-1]$. Let $S_i = \{j : Y_i \text{ and } Y_j \text{ are not independent}\}$. We observe that the Y_i are identically distributed (by symmetry), and, uniformly in i,

$$\begin{aligned} M_4 := \mathbb{E}\left[Y_i^4\right] &= \mathbb{E}\left[\left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n-1} a X_i X_{i+1} + \frac{1}{\sqrt{n}} \sum_{i=1}^{n-1} b X_{\Pi_n(i)} X_{\Pi_n(i+1)}\right)^4\right] \\ &\le 2^3 \mathbb{E}\left[\frac{a^4}{n^2} X_i^4 X_{i+1}^4 + \frac{b^4}{n^2} X_{\Pi_n(i)}^4 X_{\Pi_n(i+1)}^4\right] \\ &\le \frac{C}{n^2} \ , \end{aligned}$$

for some C depending only on a and b. Similarly, for $M_2 := \mathbb{E}Y_i^2$, $M_2 \le C/n$, for some C depending only on a and b (we may choose C to be the same for both M_2 and M_4). Also, $\mathbb{E}Y_i = 0$.

Now, condition on $\Pi_n = \pi$, where we leave the dependence of π on n implicit. In the following, expectations are taken conditional on $\Pi_n = \pi$. Consider the vectors

$$v_1 = \begin{pmatrix} X_i X_{i+1} \\ X_{\pi(i)} X_{\pi(i+1)} \end{pmatrix}, \quad v_2 = \begin{pmatrix} X_j X_{j+1} \\ X_{\pi(j)} X_{\pi(j+1)} \end{pmatrix} \ .$$

v_1 and v_2 are independent unless, for

$$\begin{aligned} A_i &:= \{i-1,\, i,\, i+1,\, i+2,\, \pi(i)-1,\, \pi(i),\, \pi(i)+1,\, \pi(i+1)-1,\, \pi(i+1),\, \pi(i+1)+1\} \\ B_j &:= \{j,\, j+1,\, \pi(j),\, \pi(j+1)\} \ , \end{aligned}$$

$A_i \cap B_j \neq \emptyset$. There are at most $|A_i|$ choices for each element of B_j, and so there are at most $|A_i|^4 \le 10^4$ vectors of the form v_2 which are not independent of v_1. It follows that, for all i, $|S_i| \le 10^4$, which is, in particular, finite.

We may now apply Theorem 2.1, noting that, by Slutsky's theorem, taking the scaling to be $\sqrt{n}$ or $\sqrt{n-1}$ is equivalent. By Remarks 2.1 and 2.2, it now suffices to show that σ_n^2, as defined in Theorem 2.1, satisfies $\sigma_n^2 = a^2 + b^2 + o(1)$. By (S.11), and the corresponding result for $\pi\mathbf{X}$,

$$\sigma_n^2 = a^2 + b^2 + o(1) + \frac{2ab}{n} \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \text{Cov}\left(X_i X_{i+1},\, X_{\pi(j)} X_{\pi(j+1)}\right) \ . \tag{S.13}$$

We claim that the sum of covariances in (S.13) is $o(1)$. To this end, we have that

$$\text{Cov}\left(X_i X_{i+1},\, X_{\pi(j)} X_{\pi(j+1)}\right) = \begin{cases} 1, & \text{if } \{\pi(j),\, \pi(j+1)\} = \{i,\, i+1\} \ , \\ 0, & \text{otherwise} \ . \end{cases}$$

Let M_n be the number of pairs of indices $(i,\, j)$ such that $\{\pi(j),\, \pi(j+1)\} = \{i,\, i+1\}$, N_n be the number of pairs of indices $(i,\, j)$ such that $i = \pi(j)$ and $i + 1 = \pi(j+1)$, and let N_n' be the number of pairs of indices $(i,\, j)$ such that $i = \pi(j+1)$ and $i + 1 = \pi(j)$. Note that $M_n = N_n + N_n'$. The following argument is taken from Ritzwoller and Romano (2020).

If M_n were uniformly bounded, it would be the case that $\sigma_n^2 = a^2 + b^2 + o(1)$. However, M_n depends on the values of the realizations π as $n \to \infty$. We argue as follows: for all n,

$$\begin{aligned} \mathbb{E} N_n &= \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbb{P}(\Pi_n(j) = i,\, \Pi_n(j+1) = i+1) \\ &= \frac{n^2}{n(n-1)} \leq 2 \ , \end{aligned}$$

and similarly for N_n'. Therefore $\{M_n,\, n \in \mathbb{N}\}$ are nonnegative random variables with uniformly bounded expectation, and so are tight. Hence, for any subsequence $\{n_j\} \subset \mathbb{N}$, there exists a further subsequence $\{n_{j_k}\}$ such that $M_{n_{j_k}}$ converges in distribution. By Skorokhod's almost sure representation theorem, there exist random variables $\tilde{M}_{n_{j_k}}$ with the same distribution as the $M_{n_{j_k}}$ which converge almost surely, to $\tilde{M}$, say.

Based on $\tilde{M} = m$, construct $\tilde{\Pi}_n$ according to the conditional distribution of $\Pi_n \,|\, M_n = m$, such that, unconditionally, $\tilde{\Pi}_n$ is uniform on S_n. Then, along a subsequence, we may use the argument above as if M were finite. In summary, given a subsequence $\{n_j\} \in \mathbb{N}$, there exists a further subsequence $n' = \{n_{j_k}\}$ such that, uniformly in t,

$$\mathbb{P}\left(\sum_{i=1}^{n_{j_k}-1} Y_i \leq t \,|\, \Pi_{n_{j_k}} \right) \xrightarrow{a.s.} \Phi\left(\frac{t}{\sqrt{a^2 + b^2}} \right) \ .$$

Therefore, unconditionally, by dominated convergence,

$$\mathbb{P}\left(\sum_{i=1}^{n_{j_k}-1} Y_i \leq t \right) \to \Phi\left(\frac{t}{\sqrt{a^2 + b^2}} \right) \ . \tag{S.14}$$

Since, given any subsequence, the result holds along a further subsequence, the limit in (S.14) holds along the original subsequence. It follows that Hoeffding's condition holds, and so the result follows. ■

The following Lemmas are required in the proof of Theorem 2.2, in the m-dependent setting.

Lemma S.2.1. *Let $X_1, \ldots, X_n$ be random variables such that $\mathbb{E}X_i = \nu_i$, $Var(X_i) = 1$, and, the X_i have uniformly bounded 4th moment.*

Let $S_i = \{j \in [n] : X_i \text{ and } X_j \text{ are not independent}\}$. Suppose $|S_i| \leq D < \infty$ for all i. Let $\sigma_n^2 = \mathbb{E}\left[\sum_{i=1}^n X_i \sum_{j \in S_i} X_j\right] = Var(\sum_{i=1}^n X_i)$, $\mu_n = \sum_{i=1}^n \nu_i$, and $W_n = \sum_{i=1}^n X_i/\sigma_n$. If $\sigma_n^2 \to 1$ and $\mu_n \to 0$, then, as $n \to \infty$,

$$\sup_{t \in \mathbb{R}} |\mathbb{P}(W_n \leq t) - \Phi(t)| \to 0 \ .$$

Proof. Let $Y_i = X_i - \nu_i$. We may now apply the result of Theorem 2.1 and Remarks 2.1 and 2.2:

$$\sup_{t \in \mathbb{R}} \left| \mathbb{P}\left(\frac{W_n - \mu_n}{\sigma_n} \leq t \right) - \Phi(t) \right| = O\left(n^{-1/4}\right) \ .$$

Reparametrizing, it follows that

$$\sup_{t \in \mathbb{R}} \left|\mathbb{P}\left(W_n \leq t\right) - \Phi\left(t\right)\right| = \sup_{t \in \mathbb{R}} \left| \mathbb{P}\left(W_n \leq t\right) - \Phi\left(\frac{t - \mu_n}{\sigma_n}\right) \right| \ .$$

Now, since $\|\cdot\|_\infty$ is a norm on $\mathbb{R}$, we have that, by the triangle inequality,

$$\begin{aligned} &\sup_{t \in \mathbb{R}} \left|\mathbb{P}\left(W_n \leq t\right) - \Phi\left(t\right)\right| \\ \leq &\sup_{t \in \mathbb{R}} \left| \mathbb{P}\left(W_n \leq t\right) - \Phi\left(\frac{t - \mu_n}{\sigma_n}\right) \right| + \sup_{t \in \mathbb{R}} \left| \Phi(t) - \Phi\left(\frac{t - \mu_n}{\sigma_n}\right) \right| \ . \end{aligned} \tag{S.15}$$

We now bound the second term in (S.15).

$$\begin{aligned} \sup_{t \in \mathbb{R}} \left| \Phi(t) - \Phi\left(\frac{t - \mu_n}{\sigma_n}\right) \right| &= \sup_{t \in \mathbb{R}} \left| \int_{(t - \mu_n)/\sigma_n}^t \phi(x) \mathrm{d}x \right| \\ &\leq \sup_{t \in \mathbb{R}} \left| t - \frac{t - \mu_n}{\sigma_n} \right| \exp\left(-\frac{1}{2} \left(t^2 \wedge \left(\frac{t - \mu_n}{\sigma_n} \right)^2 \right) \right) \\ &\to 0 \ . \end{aligned}$$

The result follows. ■

Lemma S.2.2. *Let $X_1, \ldots, X_n$ be a stationary, m-dependent time series, with mean 0, kth-order autocorrelation ρ_k, $k \in [m]$, variance $\sigma^2 = 1$, and finite 8th moment. Let $\Pi_n \sim$ Unif$\{S_n\}$. Let*

$$W_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n-1} X_{\Pi_n(i)} X_{\Pi_n(i+1)} \ .$$

Then, as $n \to \infty$,

$$\sup_{t \in \mathbb{R}} |\mathbb{P}(W_n \leq t) - \Phi(t)| \to 0 \ .$$

Proof. By application of Cauchy-Schwarz and the given properties of $\{X_i, \, i \in [n]\}$, note that $\mathbb{E}X^4_{\Pi_n(i)} X^4_{\Pi_n(i+1)}$ is uniformly bounded. In addition, conditional on $\Pi_n = \pi$ (where we leave the dependence of π on n implicit),

$$\begin{aligned} \mathbb{E}X_{\pi(i)} X_{\pi(i+1)} &= \sum_{j=1}^{n} \sum_{k \neq j} 1\{\pi(i) = j, \, \pi(i+1) = k\} \mathbb{E}X_j X_k \\ &= \sum_{j=1}^{n} \sum_{k \neq j} \sum_{l=1}^{m} \rho_l 1\{\pi(i) = j, \, \pi(i+1) = k, \, |j-k| = l\} \ . \end{aligned}$$

Let $Y_i = \frac{1}{\sqrt{n}} \left(X_{\pi(i)} X_{\pi(i+1)} - \mathbb{E}X_{\pi(i)} X_{\pi(i+1)} \right)$. Observe that, in the setting of Theorem 2.1, $|S_i| \leq 4m^2$ (since at most $2m$ of the X_j are not independent of $X_{\pi(i)}$, and similarly for $X_{\pi(i+1)}$), so we may apply Theorem 2.1 and Remarks 2.1 and 2.2: letting $Z_n = \sum_{i=1}^{n-1} Y_i / \sigma_n$, where $\sigma_n^2 = \text{Var}\left(\sum_{i=1}^{n-1} Y_i \right)$,

$$\sup_{t \in \mathbb{R}} |\mathbb{P}(Z_n \leq t) - \Phi(t)| = O(n^{-1/4})$$

Let $\mu_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n-1} \mathbb{E}X_{\pi(i)} X_{\pi(i+1)}$.

$$\sigma_n^2 = \frac{1}{n} \sum_{i=1}^{n-1} \mathbb{E}X^2_{\pi(i)} X^2_{\pi(i+1)} + \frac{1}{n} \sum_{i=1}^{n-1} \sum_{j \neq i} \text{Cov}\left(X_{\pi(i)} X_{\pi(i+1)}, \, X_{\pi(j)} X_{\pi(j+1)} \right)$$

We wish to show that $\mu_n \to 0$, and $\sigma_n^2 \to 1$. Unfortunately, this does not occur for an arbitrary $\pi \in S_n$. However, we may argue as in the proof of the i.i.d. setting of Theorem 2.2. Let K be such that, for all i, j, k and l,

$$\begin{aligned} |\mathbb{E}X_i X_j| &\leq K \\ \left|\mathbb{E}X_i^2 X_j^2\right| &\leq K \\ |\text{Cov}\left(X_i X_j, \, X_k X_l \right)| &\leq K \ . \end{aligned}$$

Such a K exists by the moment assumptions on the X_i. Note that, for $i < j < k < l$, $\text{Cov}(X_i X_j,\, X_k X_l) = 0$ unless $j \le i + m$, $l \le k + m$ (since otherwise one of the terms in the product is independent of the others). This also holds for any permutation of i, j, k, and l, by the same argument. Let

$$\begin{aligned} A_n &= \{i : |\Pi_n(i) - \Pi_n(i+1)| \le m\} \\ \Gamma_n &= \{\{i,\, j,\, k,\, l\} : i < j < k < l,\, j \le i + m,\, l \le k + m\} \\ B_n &= \{\{i,\, j\} : \{\Pi_n(i),\, \Pi_n(i+1),\, \Pi_n(j),\, \Pi_n(j+1)\} \in \Gamma_n\} \ . \end{aligned} \tag{S.16}$$

Note that (still conditional on $\Pi_n = \pi$), $\mathbb{E} X_{\pi(i)} X_{\pi(i+1)} = 0$ unless $i \in A_n$, $\mathbb{E} X^2_{\pi(i)} X^2_{\pi(i+1)} = 1$ unless $i \in A_n$, and $\text{Cov}\left(X_{\pi(i)} X_{\pi(i+1)},\, X_{\pi(j)} X_{\pi(j+1)}\right) = 0$ unless $\{i,\, j\} \in B_n$. Note that all expectations and covariances are uniformly bounded. Unconditionally, we have that

$$\begin{aligned} \mathbb{E}\,|A_n| &= \sum_{i=1}^n \sum_{j=1}^n \mathbb{P}\left(\Pi_n(i) = j,\, |\Pi_n(i+1) - j| \le m\right) \\ &\le \sum_{i=1}^n \sum_{j=1}^n \frac{1}{n} \cdot \frac{2m}{n} \\ &= 2m \ . \end{aligned} \tag{S.17}$$

Also, by assuming without loss of generality that $\Pi_n(i) < \Pi_n(i+1) < \Pi_n(j) < \Pi_n(j+1)$ and then multiplying by the appropriate number of possible permutations of these values,

$$\begin{aligned} \mathbb{E}\,|B_n| &\le 24 \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \sum_{l=1}^n \mathbb{P}\left(\Pi_n(i) = j,\, j < \Pi_n(i+1) \le j + m,\, \Pi_n(k) = l,\, l < \Pi_n(k+1) \le l + m\right) \\ &\le \frac{24}{n^2} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \sum_{l=1}^n \frac{m}{n} \cdot \frac{m}{n} \\ &= 24 m^2 \ . \end{aligned} \tag{S.18}$$

In particular, letting $M_n = |A_n| + |B_n|$, we note that $\{M_n\}$ are nonnegative random variables with uniformly bounded expectation. Hence, proceeding as in the proof of the i.i.d. setting of Theorem 2.2, the result follows, by Lemma S.2.1 (noting that scaling by $\sqrt{n}$ or $\sqrt{n-1}$ is equivalent, by Slutsky's theorem). ∎

PROOF OF THEOREM 2.2, IN THE m-DEPENDENT SETTING. By the same argument as in the proof of the i.i.d. setting of Theorem 2.2, we may assume, without loss of generality, that $\mu = 0$ and $\sigma^2 = 1$.

Note that, both $\bar{X}_n$ and $\hat{\sigma}^2_n$ are invariant under permutation. Also, by the m-dependent CLT, since $\{X_i X_{i+1},\, i \in \mathbb{N}\}$ is a stationary $(m+1)$-dependent time series with finite mean and variance,

$$\bar{X}_n = O_p\left(\frac{1}{\sqrt{n}}\right) \ .$$

Now, note that, by definition of m-dependence, for all $j \in \{1,\, \ldots,\, m{+}1\}$, $\{X_{j+k(m+1)},\, k \geq 0\}$ are independent. Hence, letting $\hat{\sigma}^2_{n,\,j} = \frac{1}{n}\sum_{j+k(m+1)\leq n} X^2_{j+k(m+1)}$, we have that, by the strong law of large numbers, for all j,

$$\hat{\sigma}^2_{n,\,j} \xrightarrow{a.s.} \frac{1}{m+1} \ .$$

Summing over j, and combining with the asymptotic distribution of $\bar{X}_n$, it follows that

$$\hat{\sigma}^2_n \xrightarrow{p} 1 \ .$$

Hence, by the same argument as in the proof of the i.i.d. setting of Theorem 2.2, it suffices to show that, for $\Pi_n,\, \Pi'_n \sim \text{Unif}\{S_n\}$, independent of each other and of the X_i (noting that $(\Pi_n,\, \Pi'_n) \stackrel{d}{=} (\Pi_n^{-1},\, (\Pi'_n)^{-1})$),

$$\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n-1} X_{\Pi_n(i)}X_{\Pi_n(i+1)},\, \frac{1}{\sqrt{n}}\sum_{i=1}^{n-1} X_{\Pi'_n(i)}X_{\Pi'_n(i+1)}\right) \xrightarrow{d} N(0,\, I_2) \ . \tag{S.19}$$

By the Cramér-Wold device, the convergence in (S.19) holds if and only if, for all $a,\, b \in \mathbb{R}$,

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n-1} aX_{\Pi_n(i)}X_{\Pi_n(i+1)} + \frac{1}{\sqrt{n}}\sum_{i=1}^{n-1} bX_{\Pi'_n(i)}X_{\Pi'_n(i+1)} \xrightarrow{d} N(0,\, a^2+b^2) \ . \tag{S.20}$$

Let $Y_i = \frac{1}{\sqrt{n}}aX_{\Pi_n(i)}X_{\Pi_n(i+1)} + \frac{1}{\sqrt{n}}bX_{\Pi'_n(i)}X_{\Pi'_n(i+1)}$, $i \in [n-1]$. Let

$$S_i = \{j : Y_i \text{ and } Y_j \text{ are not independent}\} \ .$$

Observe that the Y_i are identically distributed (by symmetry), and, uniformly in i,

$$\begin{aligned}
M_4 := \mathbb{E}\left[Y_i^4\right] &= \mathbb{E}\left[\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n-1} aX_{\Pi_n(i)}X_{\Pi_n(i+1)} + \frac{1}{\sqrt{n}}\sum_{i=1}^{n-1} bX_{\Pi'_n(i)}X_{\Pi'_n(i+1)}\right)^4\right] \\
&\leq 2^3\mathbb{E}\left[\frac{a^4}{n^2}X^4_{\Pi_n(i)}X^4_{\Pi_n(i+1)} + \frac{b^4}{n^2}X^4_{\Pi'_n(i)}X^4_{\Pi'_n(i+1)}\right] \\
&\leq \frac{C}{n^2} \ ,
\end{aligned}$$

for some C depending only on a and b. Similarly, for $M_2 := \mathbb{E}Y_i^2$, $M_2 \leq C/n$, for some C depending only on a and b (C may be chosen to be the same for both M_2 and M_4).

Now, condition on $\Pi_n = \pi$, $\Pi_n' = \pi'$, where we leave the dependence of π and π' on n implicit. In the following, expectations are taken conditional on $\Pi_n = \pi$, $\Pi_n' = \pi'$. Consider the vectors

$$v_1 = \begin{pmatrix} X_{\pi(i)} X_{\pi(i+1)} \\ X_{\pi'(i)} X_{\pi'(i+1)} \end{pmatrix}, \quad v_2 = \begin{pmatrix} X_{\pi(j)} X_{\pi(j+1)} \\ X_{\pi'(j)} X_{\pi'(j+1)} \end{pmatrix} .$$

v_1 and v_2 are independent unless, for

$$\begin{aligned} A_i &:= \bigcup_{r=0}^{1} \{k : |k - \pi(i+r)| \leq m\} \cup \bigcup_{r=0}^{1} \{k : |k - \pi'(i+r)| \leq m\} \\ B_j &:= \{\pi(j), \, \pi(j+1), \, \pi'(j), \, \pi'(j+1)\} , \end{aligned}$$

$A_i \cap B_j \neq \emptyset$. There are at most $|A_i|$ choices for each element of B_j, and so there are at most $|A_i|^4 \leq (4(2m+1))^4$ vectors of the form v_2 which are not independent of v_1. It follows that, for all i, $|S_i| \leq 256(2m+1)^4$, which is, in particular, finite.

We may now apply Theorem 2.1 to $Z_i = Y_i - \mathbb{E}Y_i$, noting that, by Slutsky's theorem, taking the scaling to be $\sqrt{n}$ or $\sqrt{n-1}$ is equivalent. By Remarks 2.1 and 2.2, it now suffices to show that σ_n^2, as defined in Theorem 2.1, satisfies $\sigma_n^2 = a^2 + b^2 + o(1)$. By Lemma S.2.2, and the corresponding result for $\pi' \mathbf{X}$,

$$\sigma_n^2 = a^2 + b^2 + o(1) + \frac{2ab}{n} \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \operatorname{Cov}\left(X_{\pi(i)} X_{\pi(i+1)}, \, X_{\pi'(j)} X_{\pi'(j+1)}\right) . \tag{S.21}$$

By the moment assumptions, we have that $\operatorname{Cov}\left(X_{\pi(i)} X_{\pi(i+1)}, \, X_{\pi'(j)} X_{\pi'(j+1)}\right)$ is uniformly bounded (by K, as defined in the proof of Lemma S.2.2).

We claim that the sum of covariances in (S.21) is $o(1)$. Note that, unless $\{\pi(i), \, \pi(i+1), \, \pi'(j), \, \pi'(j+1)\} \in \Delta_n$, where

$$\Delta_n = \{\{i, \, j, \, k, \, l\} : i \leq j \leq k \leq l, \, j \leq i + m, \, l \leq k + m\} ,$$

we will have $\operatorname{Cov}\left(X_{\pi(i)} X_{\pi(i+1)}, \, X_{\pi'(j)} X_{\pi'(j+1)}\right) = 0$, since at least one of the terms will be independent of all the others. Let

$$C_n = \{\{i, \, j\} : \{\Pi_n(i), \, \Pi_n(i+1), \, \Pi_n(j), \, \Pi_n(j+1)\} \in \Delta_n\} .$$

Conditional on Π_n, Π_n', it is the case that C_n is not necessarily finite. However, by the same argument as for B_n in the proof of Lemma S.2.2,

$$
\begin{aligned}
&\mathbb{E}\,|C_n| \\
&\leq 24 \sum_{i,\,j,\,k,\,l\in[n]} \mathbb{P}\left(\Pi_n(i)=k,\ k\leq \Pi_n(i+1)\leq k+m,\ \Pi_n'(j)=l,\ l\leq \Pi_n'(j+1)\leq l+m\right) \\
&\leq \frac{24}{n^2}\sum_{i,\,j,\,k,\,l\in[n]} \frac{m}{n}\cdot\frac{m}{n} \\
&= 24m^2\ ,
\end{aligned}
$$

and so the random variables $\{|C_n|\}$ are nonnegative and have uniformly bounded expectation. By the same argument as in the proof of the i.i.d. setting of Theorem 2.2, (S.20) holds, and so the result follows. ■

Proof of Theorem 2.3. We may assume, without loss of generality, that $\mathbb{E}X_1 = 0$. For $i \in [n-1]$, let

$$
Y_i = a\left(X_i - \bar{X}_n\right)\left(X_{i+1} - \bar{X}_n\right) + b\left(X_i - \bar{X}_n\right)^2\ .
$$

Let

$$
Z_i = aX_iX_{i+1} + bX_i^2\ .
$$

Note that

$$
Y_i = Z_i + \left[-\bar{X}_n(aX_i + 2bX_i + aX_{i+1}) + (a+b)\bar{X}_n^2\right]\ .
$$

It follows that

$$
\begin{aligned}
\sum_{i=1}^{n-1} Y_i &= \sum_{i=1}^{n-1} Z_i + \left[-\bar{X}_n\cdot 2(a+b)\left(n\bar{X}_n + O_p(1)\right) + (a+b)n\bar{X}_n^2\right] \\
&= \sum_{i=1}^{n-1} Z_i - n(a+b)\bar{X}_n^2 + o_p\left(\sqrt{n}\right)\ .
\end{aligned}
$$

By the m-dependent central limit theorem (in (i)), and Ibragimov's central limit theorem (in (ii); see Ibragimov (1962)) we have that

$$
\bar{X}_n = O_p\left(\frac{1}{\sqrt{n}}\right)\ ,
$$

and so

$$
\sum_{i=1}^{n-1} Y_i = \sum_{i=1}^{n-1} Z_i + o_p(\sqrt{n})\ .
$$

Note that the sequence $\{Z_i,\, i \in [n-1]\}$ is, in case (i), $(m+1)$-dependent, with finite second moments, by the triangle inequality and the Cauchy-Schwarz inequality, and in case (ii), is α-mixing, with α-mixing coefficients satisfying

$$\alpha_Z(i) = \alpha_X(i-1) \ ,$$

and, by the same argument as in case i), has finite $(2+\delta)$ moments. Also, in both cases, the sequence $\{Z_i\}$ is strictly stationary, and

$$\mathbb{E}[Z_i] = a\rho_1\sigma^2 + b\sigma^2 \ .$$

Also, note that

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n-1} Y_i = a\sqrt{n}\hat{\rho}_n\hat{\sigma}_n^2 + b\sqrt{n}\hat{\sigma}_n^2 + o_p(1) \ .$$

Hence, by the m-dependent central limit theorem and Ibragimov's central limit theorem and Slutsky's theorem (to rescale from $\sqrt{n-1}$ to $\sqrt{n}$), we have that

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n-1} Y_i - \sqrt{n}\left(a\rho_1\sigma^2 + b\sigma^2\right) \xrightarrow{d} N\left(0,\, \tilde{\sigma}^2\right) \ ,$$

where

$$\begin{aligned}\tilde{\sigma}^2 &= \mathrm{Var}\left(aX_1X_2 + bX_1^2\right) + 2\sum_{i\ge 2}\mathrm{Cov}\left(aX_1X_2 + bX_1^2,\, aX_iX_{i+1} + bX_i^2\right) \\ &= a^2\tau_1^2 + b^2\kappa^2 + 2ab\nu_1 \ .\end{aligned}$$

It follows that, by the Cramér-Wold device,

$$\sqrt{n}\begin{pmatrix}\hat{\rho}_n\sigma_n^2 - \rho_1\sigma^2 \\ \hat{\sigma}_n^2 - \sigma^2\end{pmatrix} \xrightarrow{d} N\left(0,\, \begin{pmatrix}\tau^2 & \nu_1 \\ \nu_1 & \kappa^2\end{pmatrix}\right) \ .$$

We may now apply the delta method, with the function $h(x,\, y) = x/y$, to conclude that

$$\sqrt{n}\,(\hat{\rho}_n - \rho) \xrightarrow{d} N(0,\, \gamma_1^2) \ ,$$

as required. ∎

Proof of Theorem 2.4. We begin by showing the result in (i). By Theorem 2.3, we have that

$$\sqrt{n}\,(\hat{\rho}_n - \rho) \xrightarrow{d} N\left(0,\, \gamma_1^2\right) \ . \tag{S.22}$$

By definition of m-dependence, it follows that $(X_1,\, X_2)$ and $(X_{i+1},\, X_{i+2})$ are independent for $i \geq m+2$. Hence

$$\begin{aligned}
\tau^2 &= \mathrm{Var}(X_1 X_2) + 2\sum_{i=1}^{m+1} \mathrm{Cov}(X_1X_2,\, X_{i+1}X_{i+2}) \\
\kappa^2 &= \mathrm{Var}\left(X_1^2\right) + 2\sum_{k=1}^{m+1} \mathrm{Cov}\left(X_1^2,\, X_k^2\right) \\
\nu_1 &= \mathrm{Cov}\left(X_1X_2,\, X_1^2\right) + \sum_{k=2}^{m+1} \mathrm{Cov}\left(X_1^2,\, X_kX_{k+1}\right) + \sum_{k=2}^{m+2} \mathrm{Cov}\left(X_1X_2,\, X_k^2\right) \ .
\end{aligned}$$

By the proof of Theorem 2.2, $\hat{\sigma}_n^2$ is a weakly consistent estimator of $\sigma^2 > 0$. Hence, by Slutsky's theorem, the continuous mapping theorem, and (S.22), it suffices to show that $\hat{\gamma}_n^2$, as defined in (2.11), is a weakly consistent estimator of γ_1^2. We may assume, without loss of generality, that $\mathbb{E}\left[X_1\right] = 0$. Let $Z_i = X_i X_{i+1}$. By the m-dependent central limit theorem, we have that $\bar{X}_n = O_p\left(1/\sqrt{n}\right)$. Hence, for all i,

$$\begin{aligned}
Y_i &= Z_i - \bar{X}_n\left(X_i + X_{i+1}\right) + \bar{X}_n^2 &&= Z_i + O_p\left(\frac{1}{\sqrt{n}}\right) \\
\bar{Y}_n &= \bar{Z}_n - \bar{X}_n^2 + o_p\left(\frac{1}{n}\right) &&= \bar{Z}_n + O_p\left(\frac{1}{n}\right) \ .
\end{aligned}$$

It follows that, for all $j \geq 0$,

$$\frac{1}{n}\sum_{i=1}^{n-j-1}\left(Y_i - \bar{Y}_n\right)\left(Y_{i+j} - \bar{Y}_n\right) = \frac{1}{n}\sum_{j=1}^{b_n}\sum_{i=1}^{n-j-1}\left(Z_i - \bar{Z}_n\right)\left(Z_{i+j} - \bar{Z}_n\right) + O_p\left(\frac{1}{\sqrt{n}}\right) \ .$$

In particular, since $b_n = o(\sqrt{n})$, we have that

$$\hat{T}_n^2 = \frac{1}{n}\sum_{i=1}^{n-1}\left(Z_i - \bar{Z}_n\right)^2 + \frac{2}{n}\sum_{j=1}^{b_n}\sum_{i=1}^{n-j-1}\left(Z_i - \bar{Z}_n\right)\left(Z_{i+j} - \bar{Z}_n\right) + o_p(1) \ .$$

In order to show consistency of $\hat{T}_n^2$, we may further assume, without loss of generality, that $\mathbb{E}[Z_i] = \mathbb{E}\left[X_1X_2\right] = 0$.

Since the sequence $\{Z_i\}$ is $(m+1)$-dependent, stationary, and has finite fourth moments, we may once again apply the m-dependent central limit theorem, as above, and conclude that

$$\begin{aligned}
\hat{T}_n^2 &= \frac{1}{n}\sum_{i=1}^{n-1} Z_i^2 + \frac{2}{n}\sum_{j=1}^{b_n}\sum_{i=1}^{n-j-1} Z_i Z_{i+j} + o_p(1) \\
&= \frac{1}{n}\sum_{i=1}^{n-1} X_i^2 X_{i+1}^2 + \frac{2}{n}\sum_{j=1}^{b_n}\sum_{i=1}^{n-j-1} X_i X_{i+1} X_{i+j} X_{i+j+1} + o_p(1) \ .
\end{aligned}$$

For each $0 \le j \le b_n$, let

$$\hat{T}_{n,j} = \frac{1}{n} \sum_{i=1}^{n-j-1} X_i X_{i+1} X_{i+j} X_{i+j+1} \ .$$

For each $j \in \mathbb{N}$, let

$$t_j = \text{Cov}\left(X_1 X_2,\, X_{j+1} X_{j+2}\right)$$

Since $b_n = o\left(\sqrt{n}\right)$, it suffices to show that, for each j,

$$\hat{T}_{n,j} - t_j = O_p\left(\frac{1}{\sqrt{n}}\right) \ . \tag{S.23}$$

Fix j. We split the sum defining $\hat{T}_{n,j}$ into $(m+1)$ sums of i.i.d. random variables, i.e. for $1 \le i \le m+1$, let $A_i = \{k \in [n] : k \equiv i \mod (m+1)\}$, and let

$$\hat{T}_{n,j,i} = \frac{1}{n} \sum_{k \in A_i,\, k+j+1 \le n} X_k X_{k+1} X_{k+j} X_{k+j+1} \ .$$

Each term in the sum is i.i.d., with mean t_j and finite variance, so, by Chebyshev's inequality, for each $i \le m+1$,

$$\hat{T}_{n,j,i} - \frac{1}{m+1} t_j = O_p\left(\frac{1}{\sqrt{n}}\right) \ .$$

Hence, summing over i, (S.23) holds, and thus $\hat{T}_n^2$ is a consistent estimator of τ_1^2. Similarly, we have that $\hat{K}_n^2$ and $\hat{\nu}_n$ are consistent estimators of κ^2 and ν_1, respectively. By Theorem 2.3, we have that $\hat{\rho}_n$ is a consistent estimator of ρ_1, and so, by several applications of Slutsky's theorem, $\hat{\gamma}_n^2$ is a consistent estimator of γ_1^2. Thus concludes the proof of part (i).

We turn our attention to part (ii) of the Theorem. Let $\Pi_n \sim \text{Unif}\{S_n\}$. By Slutsky's theorem for randomization distributions, Theorem 2.2, and the consistency of $\hat{\sigma}_n^2$ under permutation, it suffices to show that

$$\hat{\gamma}_n^2\left(\Pi_n \mathbf{X}\right) \xrightarrow{p} \sigma^4 \ . \tag{S.24}$$

By Theorem 2.2 and Hoeffding's condition (see Chung and Romano (2013), Theorem 5.1), we have that

$$\hat{\rho}_n\left(\Pi_n \mathbf{X}\right) \xrightarrow{p} 0 \ .$$

Therefore, by Slutsky's theorem, it suffices to show the following:

$$\hat{T}_n^2(\Pi_n \mathbf{X}) \xrightarrow{p} \sigma^4$$
$$\hat{K}_n^2(\Pi_n \mathbf{X}) \xrightarrow{p} \mathrm{Var}\left(X_1^2\right)$$
$$\hat{\nu}_n(\Pi_n \mathbf{X}) \xrightarrow{p} 0 \ .$$

We begin by showing that $\hat{T}_n^2(\Pi_n \mathbf{X}) \xrightarrow{p} \sigma^4$. Once again, we may assume that $\mathbb{E}X_1 = 0$. Let $\zeta_i = X_{\Pi_n(i)} X_{\Pi_n(i+1)}$. Noting that the sample mean is invariant under permutation, we may apply the same argument as in the proof of part i), and so

$$\hat{T}_n^2(\Pi_n \mathbf{X}) = \frac{1}{n} \sum_{i=1}^{n-1} \left(\zeta_i - \bar{\zeta}_n\right)^2 + \frac{2}{n} \sum_{j=1}^{b_n} \sum_{i=1}^{n-j-1} \left(\zeta_i - \bar{\zeta}_n\right)\left(\zeta_{i+j} - \bar{\zeta}_n\right) + o_p(1) \ .$$

Let

$$A_n = \{i \in [n-1] : |\Pi_n(i) - \Pi_n(i+1)| \le m\} \ .$$

Note that this A_n is the same as the one defined in (S.16). We have that

$$\hat{T}_{n,0}(\Pi_n \mathbf{X}) \xrightarrow{p} \sigma^4 \ , \tag{S.25}$$

and, for $1 \le j \le b_n$,

$$\hat{T}_{n,j}(\Pi_n \mathbf{X}) = o_p\left(\frac{1}{\sqrt{n}}\right) \ . \tag{S.26}$$

We begin by proving (S.25). Condition on $\Pi_n = \pi$. Unless $i \in A_n$, we have that $\mathbb{E}X_{\pi(i)}^2 X_{\pi(i+1)}^2 = \sigma^4$, and, if $i \in A_n$, $\mathbb{E}X_{\pi(i)}^2 X_{\pi(i+1)}^2 \le K$, for some finite K (as defined in the proof of Lemma S.2.2). Hence

$$\mathbb{E}\left[\left|\frac{1}{n}\sum_{i=1}^{n-1} X_{\Pi_n(i)}^2 X_{\Pi_n(i+1)}^2 - \sigma^4\right| \mid \Pi_n\right] \le \frac{1 + (K + \sigma^4)\,|A_n|}{n} \ . \tag{S.27}$$

Let $\epsilon > 0$. By Markov's inequality,

$$\begin{aligned} \mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^{n-1} X_{\Pi_n(i)}^2 X_{\Pi_n(i+1)}^2 - 1\right| > \epsilon\right) &\le \frac{\mathbb{E}\left|\frac{1}{n}\sum_{i=1}^{n-1} X_{\Pi_n(i)}^2 X_{\Pi_n(i+1)}^2 - 1\right|}{\epsilon} \\ &\le \frac{1 + (K + \sigma^4)\mathbb{E}\,|A_n|}{n\epsilon} \quad \text{(by (S.27))} \\ &\le \frac{1 + 2m(K + \sigma^4)}{n\epsilon} \quad \text{(by (S.17))} \\ &\to 0 \ \text{ as } n \to \infty \ . \end{aligned} \tag{S.28}$$

Hence (S.25) holds. It remains to prove (S.26). Let Γ_n be as defined in (S.16). Fix $j \in [m+1]$. Let

$$B'_n = \{\{i,\, j\} : \{\Pi_n(i),\, \Pi_n(i+1),\, \Pi_n(i+j),\, \Pi_n(i+j+1)\} \in \Gamma_n\} \ .$$

Condition once more on $\Pi_n = \pi$. We observe that, for B_n as defined in (S.16), we have that $\mathbb{E}\,|B'_n| \leq \mathbb{E}\,|B_n|$. Also note that, unless $\{i,\, j\} \in B'_n$, we have that

$$\mathbb{E} X_{\Pi_n(i)} X_{\Pi_n(i+1)} X_{\Pi_n(i+j)} X_{\Pi_n(i+j+1)} = 0 \ ,$$

and, if $\{i,\, j\} \in B'_n$, the expectation is, once again, bounded uniformly by K. Therefore, unconditionally,

$$\mathbb{E}\left|\sum_{j=1}^{n-2} \frac{1}{n} \sum_{i=1}^{n-j-1} X_{\Pi_n(i)} X_{\Pi_n(i+1)} X_{\Pi_n(i+j)} X_{\Pi_n(i+j+1)}\right| \leq \frac{K\mathbb{E}\,|B'_n|}{n} \ , \tag{S.29}$$

and, in particular,

$$\mathbb{E}\left|\frac{1}{n} \sum_{i=1}^{n-j-1} X_{\Pi_n(i)} X_{\Pi_n(i+1)} X_{\Pi_n(i+j)} X_{\Pi_n(i+j+1)}\right| \leq \frac{K\mathbb{E}\,|B'_n|}{n} \ . \tag{S.30}$$

Let $\epsilon > 0$. By Markov's inequality,

$$\begin{aligned}
&\mathbb{P}\left(\left|\frac{1}{n} \sum_{i=1}^{n-j-1} X_{\Pi_n(i)} X_{\Pi_n(i+1)} X_{\Pi_n(i+j)} X_{\Pi_n(i+j+1)}\right| > \epsilon\right) \\
&\leq \frac{\mathbb{E}\left|\frac{1}{n} \sum_{i=1}^{n-j-1} X_{\Pi_n(i)} X_{\Pi_n(i+1)} X_{\Pi_n(i+j)} X_{\Pi_n(i+j+1)}\right|}{\epsilon} \\
&\leq \frac{K\mathbb{E}\,|B'_n|}{n\epsilon} \quad \text{(by } (S.30)\text{)} \\
&\leq \frac{24Km^2}{n\epsilon} \quad \text{(by } (S.18)\text{)} \ .
\end{aligned}$$

Hence the first convergence in (S.28) holds. By an identical argument, the remaining two convergences in (S.28) also hold, and so the result follows. ∎

S.3 Permutation distribution for α-mixing sequences

Lemma S.3.1. *Let $\{X_n,\, n \in \mathbb{Z}\}$ be a stationary, α-mixing sequence. If the sequence $\{X_n\}$ is exchangeable, then the X_n are independent and identically distributed.*

Proof. It suffices to show that, for all $n \in \mathbb{N}$, for events $A_i \in \sigma(X_i)$,

$$\mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right) = \prod_{i=1}^{n} \mathbb{P}(A_i) \ .$$

We show this by induction on n. As a base case, take $n = 2$. By the exchangeability condition, we have that, for all $N > 2$,

$$\begin{aligned}
&\{Y_n,\, n \in \mathbb{Z}\} \\
&:= \{\ldots, X_{-2},\, X_{-1},\, X_2,\, X_0,\, X_N,\, X_{N-1},\, \ldots,\, X_1,\, X_{N+1},\, X_{N+2},\, \ldots\} \stackrel{d}{=} \{X_n,\, n \in \mathbb{Z}\} \ .
\end{aligned}$$

In particular, considering the α-mixing coefficients of Y, we have that

$$\begin{aligned}
|\mathbb{P}\left(A_1 \cap A_2\right) - \mathbb{P}(A_1)\mathbb{P}(A_2)| &\leq \alpha_Y(N+1) \\
&= \alpha_X(N+1) \\
&\to 0 \ , \ \text{ as } N \to \infty \ .
\end{aligned}$$

Hence the result holds for $n = 2$. We now show the inductive step, assuming the inductive hypothesis holds for all $m \leq n$. For all $1 \leq i \leq n+1$, let $A_i \in \sigma(X_i)$. Let

$$B = \cap_{i=1}^{n} A_i \ .$$

By exchangeability, we have that, for all $N > n+1$,

$$\begin{aligned}
&\{Y_n,\, n \in \mathbb{Z}\} \\
&:= \{\ldots, X_{-2},\, X_{-1},\, X_{n+1},\, X_0,\, X_N,\, X_{N-1},\, \ldots,\, X_{n+2},\, X_1,\, X_2,\, \ldots, X_n,\, X_{N+1},\, X_{N+2},\, \ldots\} \\
&\stackrel{d}{=} \{X_n,\, n \in \mathbb{Z}\} \ .
\end{aligned}$$

In particular, it follows that

$$\begin{aligned}
|\mathbb{P}\left(A_{n+1} \cap B\right) - \mathbb{P}(A_{n+1})\mathbb{P}(B)| &\leq \alpha_Y(N-n+1) \\
&= \alpha_X(N-n+1) \\
&\to 0 \ , \ \text{ as } N \to \infty \ .
\end{aligned}$$

Hence, by the inductive hypothesis,

$$\begin{aligned}
\mathbb{P}\left(\bigcap_{i=1}^{n+1} A_i\right) &= \mathbb{P}(A_{n+1})\mathbb{P}(B) \\
&= \prod_{i=1}^{n+1} \mathbb{P}(A_i) \ ,
\end{aligned}$$

as claimed. ■

Remark S.3.1. Since m-dependent sequences are α-mixing, the result of Lemma S.3.1 also holds for $\{X_n\}$ an m-dependent sequence. ■

The following Lemma, and its proof, will be used extensively to show further results throughout this Section.

Lemma S.3.2. *Let $\{X_n\}$ be a stationary, α-mixing sequence, with mean 0 and variance 1, such that*

$$\|X_n\|_{8+4\delta} < \infty \ ,$$

for some $\delta > 0$. Suppose the α-mixing coefficients of $\{X_n\}$ satisfy

$$\sum_{n \geq 1} \alpha_X(n)^{\frac{\delta}{2+\delta}} < \infty \ .$$

Let $\Pi_n \sim$ Unif(S_n), independent of $\{X_n\}$. We have that, as $n \to \infty$,

$$\mathbb{E}\left[\frac{1}{\sqrt{n}} \sum_{i=1}^{n-1} X_{\Pi_n(i)} X_{\Pi_n(i+1)}\right] \to 0 \ , \tag{S.31}$$

and

$$Var\left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n-1} X_{\Pi_n(i)} X_{\Pi_n(i+1)}\right) \to 1 \ . \tag{S.32}$$

Remark S.3.2. *For $\{X_n\}$ a stationary sequence, and for fixed $n \in \mathbb{N}$, the sequence $\{X_{\Pi_n(i)} X_{\Pi_n(i+1)},\ i \in [n-1]\}$ is also stationary.*■

Proof. We begin by showing (S.31). By Remark S.3.2, it suffices to show that, as $n \to \infty$,

$$\sqrt{n}\mathbb{E}\left[X_{\Pi_n(1)} X_{\Pi_n(2)}\right] \to 0 \ .$$

For ease of notation, let $\alpha(\cdot)$ denote $\alpha_X(\cdot)$.

$$\begin{aligned} \left|\mathbb{E}\left[X_{\Pi_n(1)} X_{\Pi_n(2)}\right]\right| &= \frac{2}{n(n-1)} \left|\sum_{i<j} \mathbb{E}\left[X_i X_j\right]\right| \\ &\leq \frac{2}{n(n-1)} \sum_{i<j} |\mathbb{E}[X_i X_j]| \\ &= \frac{2}{n(n-1)} \sum_{i<j} |\text{Cov}(X_i,\, X_j)| \ . \end{aligned}$$

By Doukhan (1994), Section 1.2.2, Theorem 3, we have that, for $1 \leq i < j \leq n$,

$$
\begin{aligned}
|\mathrm{Cov}(X_i,\, X_j)| &\leq 8\alpha(j-i)^{\frac{\delta}{2+\delta}} \|X_i\|_{2+\delta}\|X_j\|_{2+\delta} \\
&\leq C_1 \alpha(j-i)^{\frac{\delta}{2+\delta}}\ ,
\end{aligned}
$$

for some constant $C_1 \in \mathbb{R}_+$. Hence, for some constant C_2, not depending on n,

$$
\begin{aligned}
\left|\mathbb{E}\left[X_{\Pi_n(1)}X_{\Pi_n(2)}\right]\right| &\leq \frac{C_2}{n^2}\sum_{i<j}\alpha(j-i)^{\frac{\delta}{2+\delta}} \\
&= \frac{C_2}{n}\sum_{i=1}^{n-1}\frac{n-i}{n}\alpha(i)^{\frac{\delta}{2+\delta}} \\
&\leq \frac{C_2}{n}\sum_{n\geq 1}\alpha(n)^{\frac{\delta}{2+\delta}} \\
&= O\left(\frac{1}{n}\right)\ ,
\end{aligned} \tag{S.33}
$$

and so (S.31) holds. We turn our attention to (S.32). By (S.31), it suffices to show that, as $n \to \infty$,

$$
\frac{1}{n}\mathbb{E}\left[\left(\sum_{i=1}^{n-1}X_{\Pi_n(i)}X_{\Pi_n(i+1)}\right)^2\right] \to 1\ .
$$

By repeated application of Remark S.3.2,

$$
\begin{aligned}
&\frac{1}{n}\mathbb{E}\left[\left(\sum_{i=1}^{n-1}X_{\Pi_n(i)}X_{\Pi_n(i+1)}\right)^2\right] \\
&= \frac{1}{n}\sum_{i=1}^{n-1}\mathbb{E}\left[X^2_{\Pi_n(i)}X^2_{\Pi_n(i+1)}\right] + \frac{1}{n}\sum_{i=1}^{n-2}\mathbb{E}\left[X_{\Pi_n(i)}X^2_{\Pi_n(i+1)X_{\Pi_n(i+2)}}\right] + \\
&+ \frac{1}{n}\sum_{|i-j|>1}\mathbb{E}\left[X_{\Pi_n(i)}X_{\Pi_n(i+1)}X_{\Pi_n(j)}X_{\Pi_n(j+1)}\right] \\
&= \frac{n-1}{n}\mathbb{E}\left[X^2_{\Pi_n(1)}X^2_{\Pi_n(2)}\right] + \frac{n-2}{n}\mathbb{E}\left[X_{\Pi_n(1)}X^2_{\Pi_n(2)}X_{\Pi_n(3)}\right] + \\
&+ \frac{(n-2)^2}{n}\mathbb{E}\left[X_{\Pi_n(1)}X_{\Pi_n(2)}X_{\Pi_n(3)}X_{\Pi_n(4)}\right]\ .
\end{aligned} \tag{S.34}
$$

We now find the limit of each term on the right hand side of (S.34). We begin by considering the first term.

$$
\begin{aligned}
\mathbb{E}\left[X_{\Pi_n(1)}^2 X_{\Pi_n(2)}^2\right] &= \frac{2}{n(n-1)} \sum_{i<j} \left(\mathbb{E}\left[X_i^2\right]\mathbb{E}\left[X_j^2\right] + \operatorname{Cov}\left(X_i^2,\, X_j^2\right)\right) \\
&= 1 + \frac{2}{n(n-1)} \sum_{i<j} \operatorname{Cov}\left(X_i^2,\, X_j^2\right) \ .
\end{aligned}
$$

By Doukhan (1994), Section 1.2.2, Theorem 3, the bounded moment condition on the $\{X_n\}$, and the Cauchy-Schwarz inequality, there exists a constant $C \in \mathbb{R}_+$, independent of n, such that

$$
\begin{aligned}
\frac{2}{n(n-1)} \left| \sum_{i<j} \operatorname{Cov}(X_i^2,\, X_j^2) \right| &\leq \frac{2}{n(n-1)} \sum_{i<j} 8\alpha(j-i)^{\frac{\delta}{2+\delta}} \|X_i^2\|_{2+\delta} \|X_j^2\|_{2+\delta} \\
&\leq \frac{2C}{n(n-1)} \sum_{i<j} \alpha(j-i)^{\frac{\delta}{2+\delta}} \\
&= o(1) \ ,
\end{aligned}
$$

by the same argument as in (S.33). Hence

$$
\frac{n-1}{n} \mathbb{E}\left[X_{\Pi_n(1)}^2 X_{\Pi_n(2)}^2\right] = 1 + o(1) \ .
$$

We now consider the second term in (S.34).

$$
\left|\mathbb{E}\left[X_{\Pi_n(1)} X_{\Pi_n(2)}^2 X_{\Pi_n(3)}\right]\right| \leq \frac{2}{n(n-1)(n-2)} \sum_{i<k,\, j \neq i,\, k} \left|\mathbb{E}\left[X_i X_j^2 X_k\right]\right| \ .
$$

Fix i and k. Suppose $j < i$ or $j > k$; without loss of generality, suppose $j < i$. By Doukhan (1994), Section 1.2.2, Theorem 3,

$$
\begin{aligned}
\left|\mathbb{E}\left[X_i X_j^2 X_k\right]\right| &= \left|\operatorname{Cov}\left(X_i X_j^2,\, X_k\right)\right| \\
&\leq \|X_k\|_{2+\delta} \|X_i X_j^2\|_{2+\delta} \cdot 8\alpha(k-i)^{\frac{\delta}{2+\delta}} \\
&\leq 8\|X_k\|_{2+\delta} \|X_i\|_{4+2\delta}^2 \|X_j\|_{8+4\delta}^4 \alpha(k-i)^{\frac{\delta}{2+\delta}} \quad \text{(Cauchy-Schwarz)} \\
&\leq C\alpha(k-i)^{\frac{\delta}{2+\delta}} \ ,
\end{aligned}
$$

for some universal constant C, by the moment condition on the (X_n). Suppose $i < j < k$. Without loss of generality, suppose $j - i \leq k - j \iff j \leq \lfloor \frac{1}{2}(k-i) \rfloor$. By a similar argument,

$$
\begin{aligned}
\left|\mathbb{E}\left[X_i X_j^2 X_k\right]\right| &= \left|\operatorname{Cov}\left(X_i X_j^2,\, X_k\right)\right| \\
&\leq \|X_k\|_{2+\delta} \|X_i\|_{4+\delta}^2 \|X_j\|_{8+4\delta}^4 \alpha(k-j)^{\frac{\delta}{2+\delta}} \\
&\leq C\alpha(k-j)^{\frac{\delta}{2+\delta}} \ ,
\end{aligned}
$$

by the moment conditions on the $\{X_n\}$. It follows that, for some $C,\, K \in \mathbb{R}_+$, independent of n,

$$
\begin{aligned}
\left|\mathbb{E}\left[X_{\Pi_n(1)} X_{\Pi_n(2)}^2 X_{\Pi_n(3)}\right]\right| &\leq \frac{C}{n^3} \sum_{i<k} \left[(n-k+i-2)\alpha(k-i)^{\frac{\delta}{2+\delta}} + 2 \sum_{j=i+1}^{i+\lfloor \frac{1}{2}(k-i) \rfloor} \alpha(k-j)^{\frac{\delta}{2+\delta}} \right] \\
&\leq \frac{K}{n^3} \sum_{r=1}^{n-1} \left[(n-r)^2 + (n-r)(n-r-1) \right] \alpha(r)^{\frac{\delta}{2+\delta}} \\
&\leq \frac{K}{n} \sum_{r=1}^{n-1} \alpha(r)^{\frac{\delta}{2+\delta}} \\
&= o(1) \ .
\end{aligned} \tag{S.35}
$$

Hence the second term in (S.34) is $o(1)$. We conclude by examining the third term in (S.34).

$$
\begin{aligned}
\left|\mathbb{E}\left[X_{\Pi_n(1)} X_{\Pi_n(2)} X_{\Pi_n(3)} X_{\Pi_n(4)}\right]\right| &= \frac{24}{n(n-1)(n-2)(n-3)} \left| \sum_{i<j<k<l} \mathbb{E}\left[X_i X_j X_k X_l\right] \right| \\
&\leq \frac{24}{n(n-1)(n-2)(n-3)} \sum_{i<j<k<l} \left|\mathbb{E}\left[X_i X_j X_k X_l\right]\right| \ .
\end{aligned}
$$

Fix $l-i=r \geq 3$. There are $(n-r)$ such choices for $(i,\, l)$. Without loss of generality, by stationarity, we may subsequently assume $i=1$, $l=r+1$. Note that, by the same argument as for the previous two terms, for $1<j<k<r+1$, for some constant $C \in \mathbb{R}_+$ independent of n,

$$
\left|\mathbb{E}\left[X_1 X_j X_k X_{r+1}\right]\right| \leq C\alpha \left(\max\{j-1,\, r+1-k\}\right)^{\frac{\delta}{2+\delta}} \ .
$$

Fix $\max\{j-1,\, r+1-k\} = s$. There are at most $2s$ choices for $(j,\, k)$. Therefore we have that, for some constants $K,\, \tilde{K} \in \mathbb{R}_+$ independent of n,

$$
\begin{aligned}
\left|\mathbb{E}\left[X_{\Pi_n(1)} X_{\Pi_n(2)} X_{\Pi_n(3)} X_{\Pi_n(4)}\right]\right| &\leq \frac{K}{n^4} \sum_{r=3}^{n-1} (n-r) \sum_{s=1}^{r-2} 2s \cdot \alpha(s)^{\frac{\delta}{2+\delta}} \\
&\leq \frac{K}{n^4} \sum_{s=1}^{n-1} 2s(n-s)^2 \alpha(s)^{\frac{\delta}{2+\delta}} \\
&\leq \frac{\tilde{K}}{n} \sum_{s=1}^{n-1} \left(\frac{s}{n}\right) \left(\frac{n-s}{n}\right)^2 \alpha(s)^{\frac{\delta}{2+\delta}} \\
&= O\left(\frac{1}{n}\right) \ ,
\end{aligned} \tag{S.36}
$$

by the summability condition on the α-mixing coefficients. ■

The following Theorem is a slightly more restrictive version of Theorem 3.1, and its restrictions will be relaxed later.

Theorem S.3.1. *Let $\{X_n\}$ be a stationary, bounded, α-mixing sequence, with mean 0 and variance 1. Suppose that*

$$\sum_{n \geq 1} \alpha_X(n) < \infty \ .$$

The permutation distribution of $\sqrt{n}\hat{\rho}_n$ based on the test statistic $\hat{\rho}_n = \hat{\rho}(X_1, \ldots, X_n)$, with associated group of transformations S_n, the symmetric group of order n, satisfies

$$\sup_{t \in \mathbb{R}} \left| \hat{R}_n(t) - \Phi(t) \right| \xrightarrow{p} 0 \ ,$$

as $n \to \infty$, where $\Phi(t)$ is the distribution of a standard Gaussian random variable.

Proof. Suppose the sequence $\{X_n\}$ is defined on the probability space $(\Omega, \mathcal{T}, \mathbb{P})$. Let $\mathcal{F} = \sigma(X_i : i \in \mathbb{N})$. Let

$$B(m) := \frac{1}{m} \sum_{i=1}^{m} X_i^2 - \frac{1}{m^2} \left(\sum_{j=1}^{m} X_j \right)^2 \ . \tag{S.37}$$

Note that $B(m) \geq 0$ for all $m \in \mathbb{N}$. We have that

$$\begin{aligned} \mathbb{E}\left[\frac{1}{m} \sum_{j=1}^{m} X_j \right] &= 0 \ , \\ \mathbb{E}\left[\frac{1}{m} \sum_{i=1}^{m} X_i^2 \right] &= 1 \ . \end{aligned} \tag{S.38}$$

Also, by Doukhan (1994), Section 1.2.2, Theorem 3, by the boundedness of the sequence $\{X_n\}$, there exists a constant C, independent of m, such that

$$
\begin{aligned}
0 \le \frac{1}{m^2}\mathbb{E}\left[\left(\sum_{j=1}^{m} X_j\right)^2\right] &= \frac{1}{m^2}\sum_{j=1}^{m}\mathbb{E}X_j^2 + \frac{2}{m^2}\sum_{i<j}\mathbb{E}X_iX_j \\
&\le \frac{1}{m} + \frac{2}{m^2}\sum_{i<j}\left|\mathrm{Cov}\left(X_i,\, X_j\right)\right| \\
&\le \frac{1}{m} + \frac{C}{m^2}\sum_{i<j}\alpha_X(j-i) \\
&= \frac{1}{m} + \frac{C}{m^2}\sum_{j=1}^{m-1}(m-j)\alpha_X(j) \\
&= o(1) \ ,
\end{aligned}
$$

since $\alpha_X(j) \to 0$ as $j \to \infty$. By Chebyshev's inequality, it follows that

$$
\frac{1}{m}\sum_{j=1}^{m} X_j \xrightarrow{p} 0 \ . \tag{S.39}
$$

Similarly, there exists a constant K, independent of m, such that

$$
\begin{aligned}
0 \le \mathrm{Var}\left(\frac{1}{m}\sum_{i=1}^{m} X_i^2\right) &= \frac{1}{m}\mathrm{Var}\left(X_1^2\right) + 2\sum_{i<j}\mathrm{Cov}\left(X_i^2,\, X_j^2\right) \\
&\le \frac{1}{m}\mathrm{Var}\left(X_1^2\right) + \frac{2}{m^2}\sum_{i<j}\left|\mathrm{Cov}\left(X_i^2,\, X_j^2\right)\right| \\
&\le \frac{1}{m}\mathrm{Var}\left(X_1^2\right) + \frac{K}{m^2}\sum_{i<j}\alpha_X(j-i) \\
&= o(1) \ .
\end{aligned}
$$

Hence, by Chebyshev's inequality,

$$
\frac{1}{m}\sum_{i=1}^{m} X_i^2 \xrightarrow{p} 1 \ . \tag{S.40}
$$

By (S.39) and (S.40), the continuous mapping theorem, and Slutsky's theorem, we have that, as $m \to \infty$,

$$
B(m) \xrightarrow{p} 1 \ .
$$

Let $\{n_j,\, j \in \mathbb{N}\} \subset \mathbb{N}$. By an alternative characterization of convergence in probability (see Theorem 20.5 in Billingsley (1995)), we have that there exists a subsequence $\{n_{j_k}\} \subset \{n_j\}$ such that

$$B\left(n_{j_k}\right) \overset{a.s.}{\to} 1 \ .$$

Let

$$A = \left\{\omega : \lim_{k\to\infty} \inf_{l\geq k} B(n_{j_l}) > 0\right\} \ . \tag{S.41}$$

We have that $\mathbb{P}(A) = 1$. Also, the sequence $\{X_n\}$ is bounded, and, for all $\omega \in A$,

$$\lim_{k\to\infty} \inf_{l\geq k} \left\{ \frac{1}{n_{j_l}} \sum_{i=1}^{n_{j_l}} X_i(\omega)^2 - \frac{1}{n_{j_l}^2} \left(\sum_{i=1}^{n_{j_l}} X_i(\omega) \right)^2 \right\} > 0 \ .$$

Let

$$Z_{\Pi_n}(\omega) := \sum_{i=1}^{n-1} X_{\Pi_n(i)}(\omega) X_{\Pi_n(i+1)}(\omega) + X_{\Pi_n(n)}(\omega) X_{\Pi_n(1)}(\omega) \ .$$

We may now apply the theorem of Wald and Wolfowitz (1943) (see page 383)[1] to conclude that, for each $\omega \in A$,

$$\frac{Z_{\Pi_{n_{j_k}}}(\omega) - \mathbb{E}_{\Pi_{n_{j_k}}}\left[Z_{\Pi_{n_{j_k}}}(\omega)\right]}{\mathrm{Var}_{\Pi_{n_{j_k}}}\left(Z_{\Pi_{n_{j_k}}}(\omega)\right)^{1/2}} \overset{d}{\to} N(0,\,1) \ .$$

Let $t \in \mathbb{R}$. We have that, for all $\omega \in A$,

$$\bar{R}_{n_{j_k}}(t)(\omega) \to \Phi(t) \ ,$$

where

$$\bar{R}_n(t) = \frac{1}{n!} \sum_{\pi_n \in S_n} 1\left\{ Z_{\pi_n} \leq t \mathrm{Var}\left(Z_{\Pi_n} \mid \mathcal{F}\right)^{1/2} + \mathbb{E}\left[Z_{\Pi_n} \mid \mathcal{F}\right] \right\} \ . \tag{S.42}$$

In particular, it follows that, since $\mathbb{P}(A) = 1$,

$$\bar{R}_{n_{j_k}}(t) \overset{a.s.}{\to} \Phi(t) \ . \tag{S.43}$$

In conclusion, for every sequence $\{n_j\} \subset \mathbb{N}$, there exists a subsequence $\{n_{j_k}\}$ such that (S.43) holds. It follows that, for all $t \in \mathbb{R}$, as $n \to \infty$,

$$\bar{R}_n(t) \overset{p}{\to} \Phi(t) \ .$$

[1] We do not, strictly speaking, apply the theorem itself. We repeat the proof of the theorem on the subsequence (n_{j_k}).

Let

$$
\begin{aligned}
M_n &= \frac{1}{\sqrt{n}}\mathbb{E}_{\Pi_n}\left[\sum_{i=1}^{n-1} X_{\Pi_n(i)}X_{\Pi_n(i+1)} + \frac{1}{\sqrt{n}}X_{\Pi_n(n)}X_{\Pi_n(1)} \,|\, \mathcal{F}\right] - \frac{1}{\sqrt{n}}X_{\Pi_n(n)}X_{\Pi_n(1)} \ , \\
S_n &= \frac{1}{\sqrt{n}}\mathrm{Var}_{\Pi_n}\left(\sum_{i=1}^{n-1} X_{\Pi_n(i)}X_{\Pi_n(i+1)} + \frac{1}{\sqrt{n}}X_{\Pi_n(n)}X_{\Pi_n(1)} \,|\, \mathcal{F}\right)^{1/2} \ .
\end{aligned}
\tag{S.44}
$$

Note that the expectations and variances in (S.44) are conditional on the data, and the last term in the definition of M_n accounts for the difference between the circular and non-circular definitions of autocovariance. Let

$$
Y_n = \frac{1}{\sqrt{n}}\sum_{i=1}^{n-1} X_{\Pi_n(i)}X_{\Pi_n(i+1)} \ .
$$

Note that

$$
\frac{Z_{\Pi_n}(\omega) - \mathbb{E}_{\Pi_n}\left[Z_{\Pi_n}(\omega)\right]}{\mathrm{Var}_{\Pi_n}\left(Z_{\Pi_n}(\omega)\right)^{1/2}} = \frac{Y_n(\omega) - M_n(\omega)}{S_n(\omega)} \ .
$$

In order to show that

$$
\hat{R}_n(t) \xrightarrow{p} \Phi(t) \ , \tag{S.45}
$$

by dividing both sides of the inequality in the indicator function in (S.42) by $\sqrt{n}$, and by Slutsky's theorem for randomization distributions (see Theorem 5.1 and Theorem 5.2 of Chung and Romano (2013)), it suffices to show that

$$
M_n \xrightarrow{p} 0 \ , \tag{S.46}
$$

and

$$
S_n \xrightarrow{p} 1 \ . \tag{S.47}
$$

We begin by showing (S.46). Note that, since the X_n are bounded,

$$
\frac{1}{\sqrt{n}}X_{\Pi_n(n)}X_{\Pi_n(1)} = o_p(1) \ . \tag{S.48}
$$

Also,

$$
\begin{aligned}
T_n &:= \frac{1}{\sqrt{n}} \mathbb{E}_{\Pi_n} \left[\sum_{i=1}^{n-1} X_{\Pi_n(i)} X_{\Pi_n(i+1)} + \frac{1}{\sqrt{n}} X_{\Pi_n(n)} X_{\Pi_n(1)} \,|\, \mathcal{F} \right] \\
&= \sqrt{n} \mathbb{E}_{\Pi_n} \left[X_{\Pi_n(1)} X_{\Pi_n(2)} \,|\, \mathcal{F} \right] \\
&= \sqrt{n} \cdot \frac{2}{n(n-1)} \sum_{i<j} X_i X_j \ .
\end{aligned} \tag{S.49}
$$

Applying the result of Doukhan (1994), Section 1.2.2, Theorem 3, and noting that the X_n are bounded, we observe that there exists a constant C, independent of n, such that

$$
\begin{aligned}
|\mathbb{E} T_n| &\leq \frac{2}{n(n-1)} \sum_{i<j} |\mathbb{E}\left[X_i X_j\right]| \\
&\leq \frac{2C}{n(n-1)} \sum_{i<j} \alpha_X(j-i) \\
&= \frac{2C}{n(n-1)} \sum_{i=1}^{n-1} (n-i) \alpha_X(i) \\
&\leq \frac{2C}{n} \sum_{i \geq 1} \alpha_X(i) \\
&= O\left(\frac{1}{n}\right) \ ,
\end{aligned} \tag{S.50}
$$

and so $\mathbb{E} T_n \to 0$ as $n \to \infty$. We now compute the variance of T_n. Note that

$$
\begin{aligned}
T_n^2 &= \frac{1}{n(n-1)^2} \sum_{i \neq j,\, k \neq l} X_i X_j X_k X_l \\
&= \frac{1}{n(n-1)^2} \left[2 \sum_{i \neq j} X_i^2 X_j^2 + 4 \sum_{i \neq j,\, k \neq i,\, j} X_i^2 X_j X_k + 24 \sum_{i<j<k<l} X_i X_k X_k X_l \right] \ .
\end{aligned}
$$

By a similar argument to the one showing convergence of (S.34) in the proof of Lemma S.3.2, we observe that

$$
\mathbb{E} T_n^2 \to 0 \ . \tag{S.51}
$$

By (S.50) and (S.51), and a straightforward application of Chebyshev's inequality, we have that

$$
T_n \xrightarrow{p} 0 \ . \tag{S.52}
$$

By Slutsky's theorem, and combining (S.48) and (S.52), we see that (S.46) holds. We turn our attention to (S.47). Since $M_n \xrightarrow{p} 0$, it suffices to show that

$$\frac{1}{n}\mathbb{E}_{\Pi_n}\left[\left(\sum_{i=1}^{n-1} X_{\Pi_n(i)}X_{\Pi_n(i+1)} + X_{\Pi_n(n)}X_{\Pi_n(1)}\right)^2 \,|\, \mathcal{F}\right] \xrightarrow{p} 1 \; . \tag{S.53}$$

By Lemma 6 of Wald and Wolfowitz (1943) and boundedness of the $\{X_n\}$, we have that

$$\frac{1}{n}\mathbb{E}_{\Pi_n}\left[\left(\sum_{i=1}^{n-1} X_{\Pi_n(i)}X_{\Pi_n(i+1)} + X_{\Pi_n(n)}X_{\Pi_n(1)}\right)^2 \,|\, \mathcal{F}\right] = \mathbb{E}_{\Pi_n}\left[X^2_{\Pi_n(1)}X^2_{\Pi_n(2)} \,|\, \mathcal{F}\right] + o(1) \; .$$

$$\begin{aligned}\mathbb{E}_{\Pi_n}\left[X_{\Pi_n(1)}X_{\Pi_n(2)} \,|\, \mathcal{F}\right] &= \frac{2}{n(n-1)}\sum_{i<j} X_i^2 X_j^2 \\ &= 1 + \frac{2}{n(n-1)}\sum_{i<j}\left(X_i^2X_j^2 - 1\right) \; .\end{aligned}$$

By a similar argument to the one used to show (S.46), we have that

$$\frac{2}{n(n-1)}\sum_{i<j}\left(X_i^2X_j^2 - 1\right) \xrightarrow{p} 0 \; ,$$

and so (S.47) holds. Hence we have that

$$\hat{R}_n(t) \xrightarrow{p} \Phi(t) \; ,$$

as claimed. ∎

The restrictions of Theorem S.3.1 may now be relaxed in order to show the result of Theorem 3.1.

Proof of Theorem 3.1. Let Π_n, $\Pi_n' \sim \text{Unif}(S_n)$ be independent, and also independent of the sequence (X_i).Let

$$\hat{\rho}_{\Pi_n} = \frac{\sum_{i=1}^{n-1}\sum_{i=1}\left(X_{\Pi_n(i)} - \bar{X}_n\right)\left(X_{\Pi_n(i+1)} - \bar{X}_n\right)}{\hat{\sigma}_n^2} \; ,$$

and define $\hat{\rho}_{\Pi_n'}$ analogously. Without loss of generality, we may assume that $\mathbb{E}[X_1] = 0$, and $\text{Var}\,(X_1) = 1$. By Hoeffding's condition (see Chung and Romano (2013), Theorem 5.1), it suffices to show that

$$\sqrt{n}\begin{pmatrix}\hat{\rho}_{\Pi_n'} \\ \hat{\rho}_{\Pi_n'}\end{pmatrix} \xrightarrow{d} N(0,\, I_2) \; ,$$

where I_2 is the 2×2 identity matrix. Note that

$$\hat{\rho}_{\Pi_n} = \frac{1}{\hat{\sigma}_n^2}\left[\frac{1}{\sqrt{n}}\sum_{i=1}^{n-1} X_{\Pi_n(i)}X_{\Pi_n(i+1)} - 2\sqrt{n}\bar{X}_n^2 + o(1)\right] .$$

By assumption, we have that

$$\mathbb{E}\left[\bar{X}_n\right] = 0 ,$$

and, by Doukhan (1994), Section 1.2.2, Theorem 3, we have that, for some universal constant K, independent of n,

$$\begin{aligned}
\text{Var}\left(\bar{X}_n\right) &= \frac{1}{n}\text{Var}(X_1) + \frac{2}{n^2}\sum_{i<j}\text{Cov}\left(X_i,\, X_j\right) \\
&\leq O\left(\frac{1}{n}\right) + \frac{K}{n^2}\sum_{i<j}\alpha_X(j-i) \\
&= O\left(\frac{1}{n}\right) .
\end{aligned}$$

It follows that $\bar{X}_n = O_p(1/\sqrt{n})$, and so

$$\hat{\rho}_{\Pi_n} = \frac{1}{\hat{\sigma}_n^2}\left[\frac{1}{\sqrt{n}}\sum_{i=1}^{n-1} X_{\Pi_n(i)}X_{\Pi_n(i+1)} + o_p(1)\right] .$$

Obtaining an analogous result for $\hat{\rho}_{\Pi_n'}$, and noting that, by the proof of Theorem S.3.1,

$$\hat{\sigma}_n^2 \xrightarrow{p} 1 ,$$

it follows that

$$\sqrt{n}\begin{pmatrix}\hat{\rho}_{\Pi_n'}\\ \hat{\rho}_{\Pi_n'}\end{pmatrix} = \frac{1}{\sigma_n^2}\begin{pmatrix}\frac{1}{\sqrt{n}}\sum_{i=1}^{n-1} X_{\Pi_n(i)}X_{\Pi_n(i+1)}\\ \frac{1}{\sqrt{n}}\sum_{i=1}^{n-1} X_{\Pi_n(i)}X_{\Pi_n(i+1)}\end{pmatrix} + o_p(1) .$$

By Theorem S.3.1, and the converse of Chung and Romano (2013), Theorem 5.1, the result follows. ■

The proof of Lemma 3.1 follows, and this will be used in extending the result of Theorem 3.1 to the setting of Theorem 3.2.

Proof of Lemma 3.1. Enumerate $\mathbb{Q}$ as $\mathbb{Q} = \{q_n : n \in \mathbb{N}\}$. For all $N \in \mathbb{N}$, there exists $\tilde{f}(N) \in \mathbb{N}$ such that, for all $n \geq \tilde{f}(N)$ and all $j \leq N$,

$$\mathbb{P}\left(\left|G_{N,n}\left(q_j\right) - g_N(q_j)\right| \geq \frac{1}{N}\right) \leq \frac{1}{N} .$$

By letting $f(1) = \tilde{f}(1)$, and recursively defining, for $N \geq 2$,

$$f(N) = \max\left\{\tilde{f}(N),\, f(N-1)+1\right\} \ ,$$

we have that f satisfies the same property as $\tilde{f}$, and is strictly increasing. For all $n \geq f(1)$, let

$$N_n = \sup\left\{N :\ f(N) \leq n\right\} \ .$$

Note that $N_n \to \infty$ as $n \to \infty$, and that N_n is nondecreasing. Fix $j \in \mathbb{N}$. Let $r,\, s \in \mathbb{N}$. We wish to show that, for all n sufficiently large,

$$\mathbb{P}\left(|G_{N_n,n}(q_j) - g_{N_n}(q_j)| \geq \frac{1}{r}\right) \leq \frac{1}{s} \ .$$

It is a fact that, for all $N \geq \max\{j,\, r,\, s\}$, for all $n \geq f(N)$,

$$\mathbb{P}\left(|G_{N,n}(q_j) - g_{N_n}(q_j)| \geq \frac{1}{r}\right) \leq \frac{1}{s} \ .$$

Also, there exists $M \in \mathbb{N}$ such that, for all $n \geq M$,

$$N_n \geq \max\{j,\, r,\, s\} \ .$$

Since $f\left(N_n\right) \leq n$, it follows that, for all $n \geq \max\{M,\, f(1)\}$,

$$\mathbb{P}\left(|G_{N_n,n}(q_j) - g_{N_n}(q_j)| \geq \frac{1}{r}\right) \leq \frac{1}{s} \ .$$

Hence we may conclude that, for all $q \in \mathbb{Q}$, as $n \to \infty$,

$$G_{N_n,n}(q) - g_{N_n}(q) \xrightarrow{p} 0 \ .$$

By Slutsky's theorem and convergence of $g_{N_n}(t)$ for all $t \in \mathbb{R}$, it follows that

$$G_{N_n,n}(q) \xrightarrow{p} g(q) \ .$$

Since $\mathbb{Q}$ is dense in $\mathbb{R}$, g is continuous, and $G_{N_n,n}$ is nondecreasing, it follows that, for all $t \in \mathbb{R}$, as $n \to \infty$,

$$G_{N_n,n}(t) \xrightarrow{p} g(t) \ ,$$

as desired. ■

We are now in a position to prove Theorem 3.2.

Proof of Theorem 3.2. For each $N \in \mathbb{N}$, $n \in \mathbb{N}$, let

$$\begin{aligned} Y_n^{(N)} &= X_n 1\{|X_n| \leq N\} \\ Z_n^{(N)} &= X_n 1\{|X_n| > N\} \ . \end{aligned}$$

We have that

$$\begin{aligned} &\frac{1}{\sqrt{n}} \sum_{i=1}^{n-1} X_{\Pi_n(i)} X_{\Pi_n(i+1)} \\ &= \frac{1}{\sqrt{n}} \sum_{i=1}^{n-1} \left(Y_{\Pi_n(i)}^{(N)} + Z_{\Pi_n(i)}^{(N)}\right)\left(Y_{\Pi_n(i+1)}^{(N)} + Z_{\Pi_n(i+1)}^{(N)}\right) \\ &= \frac{1}{\sqrt{n}} \sum_{i=1}^{n-1} Y_{\Pi_n(i)}^{(N)} Y_{\Pi_n(i+1)}^{(N)} + \frac{1}{\sqrt{n}} \sum_{i=1}^{n-1} Z_{\Pi_n(i)}^{(N)} Y_{\Pi_n(i+1)}^{(N)} + \\ &+ \frac{1}{\sqrt{n}} \sum_{i=1}^{n-1} Y_{\Pi_n(i)}^{(N)} Z_{\Pi_n(i+1)}^{(N)} + \frac{1}{\sqrt{n}} \sum_{i=1}^{n-1} Z_{\Pi_n(i)}^{(N)} Z_{\Pi_n(i+1)}^{(N)} \ . \end{aligned} \tag{S.54}$$

Note that all of these terms are random with respect to both the sequence $\{X_n\}$ and the permutation Π_n. Let

$$\begin{aligned} \mu_N &= \mathbb{E}\left[X_1 1\{|X_1| \leq N\}\right] \ , \\ \sigma_N^2 &= \text{Var}\left(X_1 1\{|X_1| \leq N\}\right) \ , \\ \nu_N &= \mathbb{E}\left[X_1 1\{|X_1| > N\}\right] \ , \\ \tau_N^2 &= \text{Var}\left(X_1 1\{|X_1| > N\}\right) \ . \end{aligned}$$

For each $N \in \mathbb{N}$, let $\hat{R}_{N,n}$ be the permutation distribution of $\sqrt{n}\hat{\rho}_n$, based on the test statistic

$$\hat{\rho}_n = \hat{\rho}_n\left(\frac{Y_1^{(N)} - \mu_N}{\sigma_N}, \ldots, \frac{Y_n^{(N)} - \mu_N}{\sigma_N}\right) \ .$$

Note that Φ is continuous, and, for each N, $\hat{R}_{N,n}$ is nondecreasing in t. Also, by Theorem 3.1, we have that, for all $N \in \mathbb{N}$, $t \in \mathbb{R}$, as $n \to \infty$,

$$\hat{R}_{N,n}(t) \xrightarrow{p} \Phi(t) \ . \tag{S.55}$$

Hence, by Lemma 3.1, there exists a sequence $\{N_n, n \in \mathbb{N}\}$, with $N_n \to \infty$, such that

$$\hat{R}_{N_n,n}(t) \xrightarrow{p} \Phi(t) \ . \tag{S.56}$$

We now show that

$$\frac{\mu_{N_n}}{\sqrt{n}} \sum_{i=1}^{n-1} \left(Y_{\Pi_n(i)}^{(N_n)} - \mu_{N_n} \right) + \frac{\mu_{N_n}}{\sqrt{n}} \sum_{i=1}^{n-1} \left(Y_{\Pi_n(i+1)}^{(N_n)} - \mu_{N_n} \right) = o_p(1) \ . \tag{S.57}$$

Note that, by dominated convergence, as $n \to \infty$,

$$\begin{aligned} \mu_{N_n} &\to 0 \ , \\ \sigma^2_{N_n} &\to 1 \ . \end{aligned} \tag{S.58}$$

Note also that, since the second moments of the sequence $\{X_n\}$ are finite, we have that, by Chebyshev's inequality,

$$\begin{aligned} &\frac{\mu_{N_n}}{\sigma^2_{N_n}\sqrt{n}} \sum_{i=1}^{n-1} \left(Y_{\Pi_n(i)}^{(N_n)} - \mu_{N_n} \right) + \frac{\mu_{N_n}}{\sigma^2_{N_n}\sqrt{n}} \sum_{i=1}^{n-1} \left(Y_{\Pi_n(i+1)}^{(N_n)} - \mu_{N_n} \right) \\ &= \frac{2\mu_{N_n}}{\sigma^2_{N_n}\sqrt{n}} \sum_{i=1}^{n} \left(Y_i^{(N_n)} - \mu_{N_n} \right) + o_p(1) \ . \end{aligned}$$

Hence, in order for (S.57) to hold, by Slutsky's theorem, it suffices to show that

$$\frac{1}{\sigma^2_{N_n}\sqrt{n}} \sum_{i=1}^{n} \left(Y_i^{(N_n)} - \mu_{N_n} \right) = O_p(1) \ . \tag{S.59}$$

For $i \in [n]$, let

$$\xi_{n,i} = \frac{1}{\sigma_{N_n}\sqrt{n}} \left(Y_i^{(N_n)} - \mu_{N_n} \right) \ .$$

For all $n \in \mathbb{N}$, $i \in [n]$, we have that

$$\begin{aligned} \mathbb{E}\xi_{n,i} &= 0 \ , \\ \sum_{i=1}^{n} \mathbb{E}\xi^2_{n,i} &= 1 \ . \end{aligned}$$

Also, by dominated convergence, as $n \to \infty$,

$$\mathrm{Var}\left(\sum_{i=1}^{n} \xi_{n,i} \right) \to \sigma^2 := \mathbb{E}\left[X_1^2\right] + 2\sum_{k\geq 2} \mathbb{E}\left[X_1 X_k\right] \geq 0 \ .$$

Since the $\xi_{n,i}$ satisfy the Lyapunov condition

$$\begin{aligned} \mathbb{E}\left[|\xi_{n,i}|^{2+\delta} \right] &\leq \frac{\left(\|X_i\|_{2+\delta} + |\mu_{N_n}| \right)^{2+\delta}}{\sigma^{2+\delta}_{N_n} n^{1+\delta/2}} \\ &= O\left(n^{-1-\delta/2} \right) \ , \end{aligned}$$

we have that, for all $\epsilon > 0$, as $n \to \infty$,

$$\sum_{i=1}^{n} \mathbb{E}\left[\xi_{n,\,i} 1\left\{\xi_{n,\,i} > \epsilon\right\}\right] \to 0 \ .$$

Since the $\xi_{n,\,i}$ have the same α-mixing coefficients as the X_i, it follows that conditions (2.3) and (2.4) of Neumann (2013) Theorem 2.1 are satisfied, and so

$$\sum_{i=1}^{n} \xi_{n,\,i} \xrightarrow{d} N(0,\,\sigma^2) \ .$$

In particular, by Slutsky's theorem, it follows that (S.59) holds, and so (S.57) also holds.

We now show that

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n-1} Z_{\Pi_n(i)}^{(N_n)} Y_{\Pi_n(i+1)}^{(N_n)} + \sqrt{n} \mu_{N_n}^2 = o_p(1) \ . \tag{S.60}$$

Since $\mu_{N_n} + \nu_{N_n} = 0$ for all $n \in \mathbb{N}$, we have that

$$\begin{aligned} &\frac{1}{\sqrt{n}} \sum_{i=1}^{n-1} Z_{\Pi_n(i)}^{(N_n)} Y_{\Pi_n(i+1)}^{(N_n)} + \sqrt{n} \mu_{N_n}^2 \\ &= \frac{1}{\sqrt{n}} \sum_{i=1}^{n-1} \left(Z_{\Pi_n(i)}^{(N_n)} - \nu_{N_n}\right) \left(Y_{\Pi_n(i+1)}^{(N_n)} - \mu_{N_n}\right) + \frac{\nu_{N_n}}{\sqrt{n}} \sum_{i=1}^{n-1} \left(Y_{\Pi_n(i+1)}^{(N_n)} - \mu_{N_n}\right) + \\ &+ \frac{\mu_{N_n}}{\sqrt{n}} \sum_{i=1}^{n-1} \left(Z_{\Pi_n(i)}^{(N_n)} - \nu_{N_n}\right) \ . \end{aligned}$$

By (S.58), $\nu_{N_n} \to 0$ as $n \to \infty$, so by (S.59), we have that

$$\frac{\nu_{N_n}}{\sqrt{n}} \sum_{i=1}^{n-1} \left(Y_{\Pi_n(i+1)}^{(N_n)} - \mu_{N_n}\right) = o_p(1) \ . \tag{S.61}$$

By a similar argument, it follows that

$$\frac{\mu_{N_n}}{\sqrt{n}} \sum_{i=1}^{n-1} \left(Z_{\Pi_n(i)}^{(N_n)} - \nu_{N_n}\right) = o_p(1) \ . \tag{S.62}$$

We now fully utilize the moment and mixing conditions on the sequence $\{X_n\}$. By an similar proof to that of Lemma S.3.2[2], noting that, for all $r \in [1,\, 8 + 4\delta]$, as $n \to \infty$,

$$\|Z_i^{(N_n)} - \nu_{N_n}\|_r \to 0 \ ,$$

[2]The only difference in the proof is that we keep track of a multiplicative factor of $\|Z_1^{(N_n)} - \nu_{N_n}\|_r^k$ in front of each term, for some $k > 1$, $r \in [1,\, 8 + 4\delta]$.

uniformly in r, we have that

$$\mathbb{E}\left[\frac{1}{\sqrt{n}}\sum_{i=1}^{n-1}\left(Z_{\Pi_n(i)}^{(N_n)}-\nu_{N_n}\right)\left(Y_{\Pi_n(i+1)}^{(N_n)}-\mu_{N_n}\right)\right]=o(1)\ ,$$

and

$$\mathrm{Var}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n-1}\left(Z_{\Pi_n(i)}^{(N_n)}-\nu_{N_n}\right)\left(Y_{\Pi_n(i+1)}^{(N_n)}-\mu_{N_n}\right)\right)=o(1)\ .$$

By Chebyshev's inequality, it follows that

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n-1}\left(Z_{\Pi_n(i)}^{(N_n)}-\nu_{N_n}\right)\left(Y_{\Pi_n(i+1)}^{(N_n)}-\mu_{N_n}\right)=o_p(1)\ . \tag{S.63}$$

By Slutsky's theorem, combining (S.61), (S.62), and (S.63), we see that (S.60) holds.

Since, for all n, $N\in\mathbb{N}$,

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n-1}Z_{\Pi_n(i)}^{(N)}Y_{\Pi_n(i+1)}^{(N)}\overset{d}{=}\frac{1}{\sqrt{n}}\sum_{i=1}^{n-1}Y_{\Pi_n(i)}^{(N)}Z_{\Pi_n(i+1)}^{(N)}\ ,$$

we have that

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n-1}Y_{\Pi_n(i)}^{(N_n)}Z_{\Pi_n(i+1)}^{(N_n)}+\sqrt{n}\mu_{N_n}^2=o_p(1)\ . \tag{S.64}$$

Now, we consider the fourth term in (S.54). Note that

$$\begin{aligned}&\frac{1}{\sqrt{n}}\sum_{i=1}^{n-1}Z_{\Pi_n(i)}^{(N_n)}Z_{\Pi_n(i+1)}^{(N_n)}-\sqrt{n}\mu_{N_n}^2\\&=\frac{1}{\sqrt{n}}\sum_{i=1}^{n-1}\left(Z_{\Pi_n(i)}^{(N_n)}-\nu_{N_n}\right)\left(Z_{\Pi_n(i+1)}^{(N_n)}-\nu_{N_n}\right)+\frac{\nu_{N_n}}{\sqrt{n}}\sum_{i=1}^{n-1}\left(Z_{\Pi_n(i)}^{(N_n)}-\nu_{N_n}\right)+\\&+\frac{\nu_{N_n}}{\sqrt{n}}\sum_{i=1}^{n-1}\left(Z_{\Pi_n(i+1)}^{(N_n)}-\nu_{N_n}\right)\ .\end{aligned}$$

By an identical argument to the one used to show (S.60), we have that

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n-1}Z_{\Pi_n(i)}^{(N_n)}Z_{\Pi_n(i+1)}^{(N_n)}-\sqrt{n}\mu_{N_n}^2=o_p(1)\ . \tag{S.65}$$

By (S.54), we have that

$$
\begin{aligned}
&\frac{1}{\sqrt{n}}\sum_{i=1}^{n-1} X_{\Pi_n(i)}X_{\Pi_n(i+1)} \\
&= \frac{1}{\sqrt{n}}\sum_{i=1}^{n-1} Y^{(N_n)}_{\Pi_n(i)}Y^{(N_n)}_{\Pi_n(i+1)} + \frac{1}{\sqrt{n}}\sum_{i=1}^{n-1} Z^{(N_n)}_{\Pi_n(i)}Y^{(N_n)}_{\Pi_n(i+1)} + \\
&+ \frac{1}{\sqrt{n}}\sum_{i=1}^{n-1} Y^{(N_n)}_{\Pi_n(i)}Z^{(N_n)}_{\Pi_n(i+1)} + \frac{1}{\sqrt{n}}\sum_{i=1}^{n-1} Z^{(N_n)}_{\Pi_n(i)}Z^{(N_n)}_{\Pi_n(i+1)} \ ,
\end{aligned}
$$

By (S.57),

$$
\begin{aligned}
&\frac{1}{\sqrt{n}}\sum_{i=1}^{n-1} Y^{(N_n)}_{\Pi_n(i)}Y^{(N_n)}_{\Pi_n(i+1)} \\
&= \frac{\sigma^2_{N_n}}{\sqrt{n}}\sum_{i=1}^{n-1}\left(\frac{Y^{(N_n)}_{\Pi_n(i)}-\mu_{N_n}}{\sigma_{N_n}}\right)\left(\frac{Y^{(N_n)}_{\Pi_n(i+1)}-\mu_{N_n}}{\sigma_{N_n}}\right) + \frac{\mu_{N_n}}{\sqrt{n}}\sum_{i=1}^{n-1}\left(Y^{(N_n)}_{\Pi_n(i)}-\mu_{N_n}\right) + \\
&+ \frac{\mu_{N_n}}{\sqrt{n}}\sum_{i=1}^{n-1}\left(Y^{(N_n)}_{\Pi_n(i+1)}-\mu_{N_n}\right) + \sqrt{n}\mu^2_{N_n} \\
&= \frac{\sigma^2_{N_n}}{\sqrt{n}}\sum_{i=1}^{n-1}\left(\frac{Y^{(N_n)}_{\Pi_n(i)}-\mu_{N_n}}{\sigma_{N_n}}\right)\left(\frac{Y^{(N_n)}_{\Pi_n(i+1)}-\mu_{N_n}}{\sigma_{N_n}}\right) + \sqrt{n}\mu^2_{N_n} + o_p(1) \ .
\end{aligned}
\tag{S.66}
$$

Hence, substituting the results of (S.60), (S.64), and (S.65) into (S.66),

$$
\frac{1}{\sqrt{n}}\sum_{i=1}^{n-1} X_{\Pi_n(i)}X_{\Pi_n(i+1)} = \frac{\sigma^2_{N_n}}{\sqrt{n}}\sum_{i=1}^{n-1}\left(\frac{Y^{(N_n)}_{\Pi_n(i)}-\mu_{N_n}}{\sigma_{N_n}}\right)\left(\frac{Y^{(N_n)}_{\Pi_n(i+1)}-\mu_{N_n}}{\sigma_{N_n}}\right) + o_p(1) \ .
$$

By (S.56), (S.58), and Slutsky's theorem for randomization distributions (Chung and Romano (2013), Theorem 5.2), it follows that, for all $t \in \mathbb{R}$, as $n \to \infty$,

$$
\hat{R}_n(t) \xrightarrow{p} \Phi(t) \ ,
$$

as claimed. ■

The following results show the consistency of the variance estimators of γ_1^2.

Lemma S.3.3. *Let $\{X_n, \, n \in \mathbb{N}\}$ be a stationary, α-mixing sequence, with mean μ, such that, for some $\delta > 0$,*

$$
\|X_1\|_{4+2\delta} < \infty \ ,
$$

and

$$\sum_{n\geq 1} \alpha_X(n)^{\frac{\delta}{2+\delta}} < \infty \ .$$

Let

$$\tau^2 = Var(X_1) + 2\sum_{k\geq 2} Cov(X_1,\, X_k) \ .$$

Suppose that $\tau^2 > 0$. *Let* $\{b_n,\, n\in\mathbb{N}\}$ *be such that, for all* n, $b_n \leq n-1$, $b_n = o(\sqrt{n})$, *and* $b_n \to \infty$ *as* $n\to\infty$. *Let*

$$\hat{\tau}_n^2 = \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \bar{X}_n\right)^2 + \frac{2}{n}\sum_{j=1}^{b_n}\sum_{i=1}^{n-j}\left(X_i - \bar{X}_n\right)\left(X_{i+j} - \bar{X}_n\right) \ .$$

We have that, as $n\to\infty$,

$$\hat{\tau}_n^2 \xrightarrow{p} \tau^2 \ .$$

Proof. Let

$$\tilde{\tau}_n^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2 + \frac{2}{n}\sum_{j=1}^{b_n}\sum_{i=1}^{n-j}(X_i-\mu)(X_{i+j}-\mu) \ .$$

We have that

$$\hat{\tau}_n^2 = \tilde{\tau}_n^2 + \left(\frac{2}{n}\sum_{j=1}^{b_n}(n-j) - 1\right)\left(\bar{X}_n - \mu\right)^2 - \frac{2\left(\bar{X}_n-\mu\right)}{n}\sum_{j=1}^{b_n}\sum_{i=1}^{n-j}\left[(X_i-\mu)+(X_{i+j}-\mu)\right] \ .$$

By Doukhan (1994), Section 1.2.2, Theorem 3, there exists a constant $C\in\mathbb{R}_+$ such that

$$\begin{aligned}
\mathrm{Var}\left(\bar{X}_n\right) &= \frac{1}{n}\mathrm{Var}(X_1) + \frac{1}{n^2}\sum_{i<j}\mathrm{Cov}\left(X_i,\, X_j\right) \\
&\leq O\left(\frac{1}{n}\right) + \frac{C}{n^2}\sum_{i<j}\alpha_X(j-i)^{\frac{\delta}{2+\delta}} \qquad\qquad \text{(S.67)} \\
&= O\left(\frac{1}{n}\right) \ ,
\end{aligned}$$

by the same argument as in (S.33). Hence, since $\mathbb{E}\bar{X}_n = \mu$,

$$\bar{X}_n - \mu = O_p\left(\frac{1}{\sqrt{n}}\right) \ ,$$

and so

$$\tilde{\tau}_n^2 = \hat{\tau}_n^2 - \frac{2\left(\bar{X}_n - \mu\right)}{n} \sum_{j=1}^{b_n} \sum_{i=1}^{n-j} \left[(X_i - \mu) + (X_{i+j} - \mu)\right] + o_p(1) \ . \tag{S.68}$$

Note that

$$\begin{aligned} \sum_{j=1}^{b_n} \sum_{i=1}^{n-j} \left[(X_i - \mu) + (X_{i+j} - \mu)\right] &= \sum_{j=1}^{b_n} \left(2n\left(\bar{X}_n - \mu\right) - \sum_{i=1}^{n-j-1} (X_i - \mu) - \sum_{r=n-j+1}^{n} (X_r - \mu) \right) \\ &= 2nb_n\left(\bar{X}_n - \mu\right) - \\ &- \sum_{i=1}^{n} \left[(b_n \wedge (n-i-1))_+ + (b_n + i - n)_+\right](X_i - \mu) \ . \end{aligned} \tag{S.69}$$

By another application of Doukhan (1994), Section 1.2.2, Theorem 3 and Chebyshev's inequality, we obtain that

$$\sum_{i=1}^{n} \left[(b_n \wedge (n-i-1))_+ + (b_n + i - (n-1))_+\right](X_i - \mu) = O_p\left(b_n\sqrt{n}\right) \ . \tag{S.70}$$

Combining (S.68), (S.69), and (S.70), we have that

$$\hat{\tau}_n^2 = \tilde{\tau}_n^2 + o_p(1) \ . \tag{S.71}$$

Therefore, assuming, without loss of generality, that $\mu = 0$, it suffices to show that

$$\tilde{\tau}_n^2 \xrightarrow{p} \tau^2 \ .$$

Fix $0 \leq j \leq b_n$. By stationarity, we have that

$$\begin{aligned} \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n-j} X_i X_{i+j}\right] &= \mathbb{E}X_1X_{1+j} - \frac{j}{n}\mathbb{E}X_1X_{1+j} \\ &= \mathbb{E}X_1X_j - O\left(\frac{1}{n}\right) \ . \end{aligned} \tag{S.72}$$

By Doukhan (1994), Section 1.2.2, Theorem 3, we also have that, for some constant $C \in \mathbb{R}_+$,

$$
\begin{aligned}
\text{Var}\left(\frac{1}{n}\sum_{i=1}^{n-j} X_i X_{i+j}\right) &= \frac{n-j}{n}\text{Var}\left(X_1 X_{1+j}\right) + \frac{2}{n^2}\sum_{1\le i<k\le n-j}\text{Cov}\left(X_i X_{i+j},\, X_k X_{k+j}\right) \\
&\le O\left(\frac{1}{n}\right) + \frac{2}{n^2}\sum_{1\le i<k\le n-j}\left|\text{Cov}\left(X_i X_{i+j},\, X_k X_{k+j}\right)\right| \\
&\le O\left(\frac{1}{n}\right) + \frac{C}{n^2}\sum_{1\le i<k\le n-j}\alpha_X\left((k-j-i)_+\right)^{\frac{\delta}{2+\delta}} \\
&= O\left(\frac{1}{n}\right) ,
\end{aligned}
$$

as in (S.33). In particular, by Chebyshev's inequality, it follows that

$$
\frac{1}{n}\sum_{i=1}^{n-j} X_i X_{i+j} = \mathbb{E} X_1 X_{1+j} + O_p\left(\frac{1}{\sqrt{n}}\right) . \tag{S.73}
$$

Hence

$$
\tilde{\tau}_n^2 = \tau^2 + O_p\left(\frac{b_n}{\sqrt{n}}\right) ,
$$

and the result follows. ■

Corollary S.3.1. *Let $\{X_n\}$ be a stationary, α-mixing sequence such that, for some $\delta > 0$,*

$$
\|X_1\|_{8+4\delta} < \infty ,
$$

and

$$
\sum_{n\ge 1}\alpha_X(n)^{\frac{\delta}{2+\delta}} < \infty .
$$

For $i \in \mathbb{N}$, let $Y_i = \left(X_i - \bar{X}_n\right)\left(X_{i+1} - \bar{X}_n\right)$. Let $b_n = o(\sqrt{n})$ be such that, as $n \to \infty$, $b_n \to \infty$. Let

$$
\hat{T}_n^2 = \frac{1}{n}\sum_{i=1}^{n-1}\left(Y_i - \bar{Y}_n\right)^2 + \frac{2}{n}\sum_{j=1}^{b_n}\sum_{i=1}^{n-j-1}\left(Y_i - \bar{Y}_n\right)\left(Y_{i+j} - \bar{Y}_n\right) .
$$

Let

$$
\tau_1^2 = Var(X_1 X_2) + 2\sum_{k\ge 2} Cov\left(X_1 X_2,\, X_k X_{k+1}\right) .
$$

We have that, as $n \to \infty$,

$$\hat{T}_n^2 \xrightarrow{p} \tau_1^2 \ .$$

Proof. Without loss of generality, we may assume that $\mathbb{E}X_1 = 0$. Let $Z_i = X_i X_{i+1}$. Let $j \geq 0$. Noting that, by the same argument as in (S.67), $\bar{X}_n = O_p(1/\sqrt{n})$, we have that

$$\begin{aligned} Y_i &= Z_i - \bar{X}_n \left(X_i + X_{i+1}\right) + \bar{X}_n^2 &&= Z_i + O_p\left(\frac{1}{\sqrt{n}}\right) \\ \bar{Y}_n &= \bar{Z}_n - \bar{X}_n^2 + o_p\left(\frac{1}{n}\right) &&= \bar{Z}_n + O_p\left(\frac{1}{n}\right) \ . \end{aligned}$$

Hence

$$\frac{1}{n}\sum_{i=1}^{n-j-1}\left(Y_i - \bar{Y}_n\right)\left(Y_{i+j} - \bar{Y}_n\right) = \frac{1}{n}\sum_{i=1}^{n-j-1}\left(Z_i - \bar{Z}_n\right)\left(Z_{i+j} - \bar{Z}_n\right) + O_p\left(\frac{1}{\sqrt{n}}\right) \ .$$

In particular, since $b_n = o(\sqrt{n})$, we have that

$$\hat{T}_n^2 = \frac{1}{n}\sum_{i=1}^{n-1}\left(Z_i - \bar{Z}_n\right)^2 + \frac{2}{n}\sum_{j=1}^{b_n}\sum_{i=1}^{n-j-1}\left(Z_i - \bar{Z}_n\right)\left(Z_{i+j} - \bar{Z}_n\right) + o_p(1) \ .$$

Since $\alpha_Z(i) = \alpha_X(i-1)$, we may apply the result of Lemma S.3.3, and the result follows. ■

Having shown consistency of the estimator $\hat{T}_n^2$, the consistency of estimators of the other terms in the definition of γ_1^2 follow similarly.

Proof of Lemma 3.2. By an identical argument to that used in the proof of Lemma S.3.3 and Corollary S.3.1, we have that

$$\begin{aligned} \hat{K}_n^2 &\xrightarrow{p} \kappa^2 \\ \hat{\nu}_n &\xrightarrow{p} \nu_1 \ . \end{aligned}$$

As a consequence of Theorem 2.3, it follows that $\hat{\rho}_n$ is a consistent estimator of ρ_1. Hence, by the continuous mapping theorem and Slutsky's theorem, we have that

$$\hat{\gamma}_n^2 \xrightarrow{p} \gamma_1^2 \ ,$$

as required. ■

We now proceed to show that, when a random permutation is applied, the studentizing factor converges in probability to 1.

Lemma S.3.4. *In the setting of Corollary S.3.1, let $\Pi_n \sim$ Unif(S_n), independent of the sequence $\{X_i,\, i \in \mathbb{N}\}$. For $i \in \mathbb{N}$, let*

$$Z_i = \left(X_{\Pi_n(i)} - \bar{X}_n\right)\left(X_{\Pi_n(i+1)} - \bar{X}_n\right) \ .$$

Let

$$\hat{T}_n^2 = \frac{1}{n}\sum_{i=1}^{n-1}\left(Z_i - \bar{Z}_n\right)^2 + \frac{2}{n}\sum_{j=1}^{b_n}\sum_{i=1}^{n-j-1}\left(Z_i - \bar{Z}_n\right)\left(Z_{i+j} - \bar{Z}_n\right) \ .$$

Then, as $n \to \infty$,

$$\hat{T}_n^2 \xrightarrow{p} Var(X_1)^2 \ .$$

Proof. We may assume, without loss of generality, that $\mathbb{E}X_1 = 0$. Let

$$\Gamma_i = X_{\Pi_n(i)} X_{\Pi_n(i+1)} \ .$$

By the same argument as in the proof of Corollary S.3.1,

$$\hat{T}_n^2 = \frac{1}{n}\sum_{i=1}^{n}\left(\Gamma_i - \bar{\Gamma}_n\right)^2 + \frac{2}{n}\sum_{j=1}^{b_n}\sum_{i=1}^{n-j-1}\left(\Gamma_i - \bar{\Gamma}_n\right)\left(\Gamma_{i+j} - \bar{\Gamma}_n\right) + o_p(1) \ .$$

By Lemma S.3.2, we have that $\bar{\Gamma}_n = O_p\left(1/\sqrt{n}\right)$. Let

$$\tilde{T}_n^2 = \sum_{i=1}^{n}\Gamma_i^2 + \frac{2}{n}\sum_{j=1}^{b_n}\sum_{i=1}^{n-j-1}\Gamma_i\Gamma_{i+j} \ .$$

Since $b_n = o\left(\sqrt{n}\right)$, we have that

$$\hat{T}_n^2 = \tilde{T}_n^2 + o_p(1) \ .$$

It therefore suffices to show that

$$\tilde{T}_n^2 \xrightarrow{p} \text{Var}\left(X_1\right)^2 \ .$$

Dividing $\tilde{T}_n^2$ by $\text{Var}\left(X_1\right)^2$, we may assume, without loss of generality, that $\text{Var}\left(X_1\right) = 1$, and subsequently it suffices to show that

$$\tilde{T}_n^2 \xrightarrow{p} 1 \ .$$

We have

$$\tilde{T}_n^2 = \frac{1}{n}\sum_{i=1}^{n-1} X_{\Pi_n(i)}^2 X_{\Pi_n(i+1)}^2 + \frac{2}{n}\sum_{j=1}^{b_n}\sum_{i=1}^{n-j-1} X_{\Pi_n(i)}X_{\Pi_n(i+1)}X_{\Pi_n(i+j)}X_{\Pi_n(i+j+1)} \ . \tag{S.74}$$

Note that, applying the result of Lemma S.3.2 to the sequence $\{X_i^2 - 1,\, i \in \mathbb{N}\}$, and by the same argument as in (S.72), we have that

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n-1} X_{\Pi_n(i)}^2 X_{\Pi_n(i+1)}^2 &= -1 + \frac{2}{n}\sum_{i=1}^{n} X_i^2 + \frac{1}{n}\sum_{i=1}^{n-1}\left(X_{\Pi_n(i)}^2 - 1\right)\left(X_{\Pi_n(i+1)}^2 - 1\right) + o_p(1) \\ &= 1 + o_p(1) \ .\end{aligned}$$

In particular, combining with (S.74), since $b_n = o(\sqrt{n})$, it suffices to show that, for each $1 \leq j \leq b_n$,

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n-j-1} X_{\Pi_n(i)}X_{\Pi_n(i+1)}X_{\Pi_n(i+j)}X_{\Pi_n(i+j+1)} = O_p\left(1\right) \ .$$

For $j = 1$, for all i, we have that

$$X_{\Pi_n(i)}X_{\Pi_n(i+1)}X_{\Pi_n(i+j)}X_{\Pi_n(i+j+1)} \stackrel{d}{=} X_{\Pi_n(1)}X_{\Pi_n(2)}^2 X_{\Pi_n(3)} \ .$$

By (S.35), we have that, for some constant $K \in \mathbb{R}_+$,

$$\begin{aligned}\left|\mathbb{E}X_{\Pi_n(1)}X_{\Pi_n(2)}^2 X_{\Pi_n(3)}\right| &\leq \frac{K}{n}\sum_{r=1}^{n-1}\alpha_X(r)^{\frac{\delta}{2+\delta}} \\ &= O\left(\frac{1}{n}\right) \ .\end{aligned} \tag{S.75}$$

It follows that

$$\mathbb{E}\left[\frac{1}{\sqrt{n}}\sum_{i=1}^{n-2} X_{\Pi_n(i)}X_{\Pi_n(i+1)}^2 X_{\Pi_n(i+2)}\right] = o(1) \ .$$

We also have that, due to the moment conditions on the sequence $\{X_n\}$ and the Cauchy-Schwarz inequality,

$$
\begin{aligned}
&\operatorname{Var}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n-2} X_{\Pi_n(i)}X^2_{\Pi_n(i+1)}X_{\Pi_n(i+2)}\right)\\
&= \frac{1}{n}\sum_{i=1}^{n-2}\operatorname{Var}\left(X_{\Pi_n(i)}X^2_{\Pi_n(i+1)}X_{\Pi_n(i+2)}\right) + \frac{2}{n}\sum_{i<j}\operatorname{Cov}\left(X_{\Pi_n(i)}X^2_{\Pi_n(i+1)}X_{\Pi_n(i+2)},\, X_{\Pi_n(j)}X^2_{\Pi_n(j+1)}X_{\Pi_n(j+2)}\right)\\
&= \frac{1}{n}\sum_{i<j-2}\operatorname{Cov}\left(X_{\Pi_n(i)}X^2_{\Pi_n(i+1)}X_{\Pi_n(i+2)},\, X_{\Pi_n(j)}X^2_{\Pi_n(j+1)}X_{\Pi_n(j+2)}\right) + O(1)\ .
\end{aligned}
$$

Note that, for all $i < j - 2$, the covariances in the above sum are equal. Therefore, in order to conclude the proof, by Chebyshev's inequality it suffices to show that

$$
\operatorname{Cov}\left(X_{\Pi_n(i)}X^2_{\Pi_n(i+1)}X_{\Pi_n(i+2)},\, X_{\Pi_n(j)}X^2_{\Pi_n(j+1)}X_{\Pi_n(j+2)}\right) = O\left(\frac{1}{n}\right)\ . \tag{S.76}
$$

By (S.75), it suffices to show that, for $i < j - 2$,

$$
\mathbb{E}\left[X_{\Pi_n(i)}X^2_{\Pi_n(i+1)}X_{\Pi_n(i+2)}X_{\Pi_n(j)}X^2_{\Pi_n(j+1)}X_{\Pi_n(j+2)}\right] = O\left(\frac{1}{n}\right)\ .
$$

There exist constants $C_1,\, C_2,\, C_3,\, C_4,\, C_5 \in \mathbb{R}_+$ such that

$$
\begin{aligned}
&\mathbb{E}\left[X_{\Pi_n(i)}X^2_{\Pi_n(i+1)}X_{\Pi_n(i+2)}X_{\Pi_n(j)}X^2_{\Pi_n(j+1)}X_{\Pi_n(j+2)}\right] = \frac{C_1}{n^6}\sum_{i,\,j,\,k,\,l,\,m,\,p\ \text{dist.}}\mathbb{E}\left[X_iX_jX_kX_lX_m^2X_p^2\right]\\
&= \frac{C_2}{n^6}\sum_{m<i<j<k<l<p}\mathbb{E}\left[X_iX_jX_kX_lX_m^2X_p^2\right] + \frac{C_3}{n^6}\sum_{i<j<k<l<m,\, i<p<m}\mathbb{E}\left[X_iX_jX_kX_lX_m^2X_p^2\right] +\\
&+ \frac{C_4}{n^6}\sum_{m<i<j<k<l,\, m<p<l}\mathbb{E}\left[X_iX_jX_kX_lX_m^2X_p^2\right] + \frac{C_5}{n^6}\sum_{i<j<k<l,\, i<m<l,\, i<p<l}\mathbb{E}\left[X_iX_jX_kX_lX_m^2X_p^2\right]\ .
\end{aligned}
\tag{S.77}
$$

We begin by considering the fourth sum on the right hand side of (S.77). Fix $l - i = r \geq 5$. There are $(n - r)$ such choices for $(i,\, l)$. By stationarity, we may assume, without loss of generality, that $m = 1$, $p = r + 1$. Assume, without loss of generality, that $m < p$. By Doukhan (1994), Section 1.2.2, Theorem 3, there exists a constant $K \in \mathbb{R}_+$ such that

$$
\left|\mathbb{E}\left[X_iX_jX_kX_lX_m^2X_p^2\right]\right| \leq K\alpha_X\left(\max\{\min\{j,\, m\} - 1,\, l - \max\{k,\, p\}\}\right)^{\frac{\delta}{2+\delta}}\ .
$$

Fix $s = \max\{\min\{j,\, m\} - 1,\, l - \max\{k,\, p\}\}$. There are, at most, $2s$ choices for the second smallest and second largest index combined, and there are at most $(n - s)^2$ choices for the remaining two indices. Hence

$$\frac{1}{n^6} \sum_{i<j<k<l,\, i<m<l,\, i<p<l} \left|\mathbb{E}\left[X_i X_j X_k X_l X_m^2 X_p^2\right]\right| \leq \frac{1}{n^6} \sum_{r=5}^{n-1} (n-r) \sum_{s=1}^{r} 2s(n-s)^2 \alpha_X(s)^{\frac{\delta}{2+\delta}}$$
$$\leq \frac{2}{n^6} \sum_{s=1}^{n-1} s(n-s)^4 \alpha_X(s)^{\frac{\delta}{2+\delta}}$$
$$= O\left(\frac{1}{n}\right) .$$

We now consider the third term on the right hand side of (S.77). Fix $l - \max\{p, k\} = r$. There are at most $n-r$ choices for the pair $(\max\{p, k\}, l)$. Now, there are at most $C \cdot (n-r)^4$ choices for the remaining indices, for some universal constant $C \in \mathbb{R}_+$. Hence, once more applying Doukhan (1994), Section 1.2.2, Theorem 3, we have that

$$\frac{1}{n^6} \sum_{m<i<j<k<l,\, m<p<l} \left|\mathbb{E}\left[X_i X_j X_k X_l X_m^2 X_p^2\right]\right| \leq \frac{C}{n^6} \sum_{r=1}^{n-1} (n-r)^5 \alpha_X(r)^{\frac{\delta}{2+\delta}}$$
$$= O\left(\frac{1}{n}\right) .$$

Similarly, the second term in (S.77) is also $O(1/n)$.

We conclude our analysis for $j = 1$ by considering the first sum in (S.77). A double application of Doukhan (1994), Section 1.2.2, Theorem 3, and the same argument as for the previous sums, gives us that, for some universal constant $C \in \mathbb{R}_+$,

$$\frac{1}{n^6} \sum_{m<i<j<k<l<p} \left|\mathbb{E}\left[X_i X_j X_k X_l X_m^2 X_p^2\right]\right|$$
$$\leq \frac{C}{n^6} \sum_{r=1}^{n-1} (n-r)^5 \alpha_X(r)^{\frac{\delta}{2+\delta}} + \frac{1}{n^6} \sum_{m<i<j<k<l<p} \left|\mathbb{E}\left[X_i X_j X_k X_l X_p^2\right]\right|$$
$$\leq O\left(\frac{1}{n}\right) + \frac{C}{n^5} \sum_{r=1}^{n-1} (n-r)^4 \alpha_X(r)^{\frac{\delta}{2+\delta}}$$
$$= O\left(\frac{1}{n}\right) .$$

It follows that (S.76) holds, by Chebyshev's inequality. We turn our attention to the corresponding result for $j > 1$. Note that, for all i, for all $j > 1$, $X_{\Pi_n(i)} X_{\Pi_n(i+1)} X_{\Pi_n(i+j)} X_{\Pi_n(i+j+1)} \stackrel{d}{=} X_{\Pi_n(i)} X_{\Pi_n(2)} X_{\Pi_n(3)} X_{\Pi_n(4)}$. By (S.36), we have that

$$\mathbb{E}\left[\frac{1}{\sqrt{n}} \sum_{i=1}^{n-j-1} X_{\Pi_n(i)} X_{\Pi_n(i+1)} X_{\Pi_n(i+j)} X_{\Pi_n(i+j+1)}\right] = o(1) . \tag{S.78}$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned}
&\mathrm{Var}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n-j-1} X_{\Pi_n(i)}X_{\Pi_n(i+1)}X_{\Pi_n(i+j)}X_{\Pi_n(i+j+1)}\right) = \mathrm{Var}\left(X_{\Pi_n(1)}X_{\Pi_n(2)}X_{\Pi_n(3)}X_{\Pi_n(4)}\right) + \\
&+ \frac{1}{n}\sum_{i\neq k}\mathrm{Cov}\left(X_{\Pi_n(i)}X_{\Pi_n(i+1)}X_{\Pi_n(i+j)}X_{\Pi_n(i+j+1)},\, X_{\Pi_n(k)}X_{\Pi_n(k+1)}X_{\Pi_n(k+j)}X_{\Pi_n(k+j+1)}\right) \\
&= \frac{1}{n}\sum_{i\in\{k\pm 1,\, k\pm j,\, k\pm(j+1)\}^C}\mathrm{Cov}\left(X_{\Pi_n(i)}X_{\Pi_n(i+1)}X_{\Pi_n(i+j)}X_{\Pi_n(i+j+1)},\, X_{\Pi_n(k)}X_{\Pi_n(k+1)}X_{\Pi_n(k+j)}X_{\Pi_n(k+j+1)}\right) + \\
&+ O(1)\ .
\end{aligned} \tag{S.79}$$

By stationarity of the sequence $\{X_i\}$ and Chebyshev's inequality, in order to conclude the proof, it suffices to show that

$$\mathrm{Cov}\left(X_{\Pi_n(1)}X_{\Pi_n(2)}X_{\Pi_n(3)}X_{\Pi_n(4)},\, X_{\Pi_n(5)}X_{\Pi_n(6)}X_{\Pi_n(7)}X_{\Pi_n(8)}\right) = O\left(\frac{1}{n}\right)\ ,$$

since the covariances in the sum of the last line of (S.79) are equal. By (S.78), it therefore suffices to show that

$$\mathbb{E}\left[X_{\Pi_n(1)}X_{\Pi_n(2)}X_{\Pi_n(3)}X_{\Pi_n(4)}X_{\Pi_n(5)}X_{\Pi_n(6)}X_{\Pi_n(7)}X_{\Pi_n(8)}\right] = O\left(\frac{1}{n}\right)\ .$$

There exists a universal constant $C \in \mathbb{R}_+$ such that

$$\begin{aligned}
&\left|\mathbb{E}\left[X_{\Pi_n(1)}X_{\Pi_n(2)}X_{\Pi_n(3)}X_{\Pi_n(4)}X_{\Pi_n(5)}X_{\Pi_n(6)}X_{\Pi_n(7)}X_{\Pi_n(8)}\right]\right| \\
&\leq \frac{C}{n^8}\sum_{i<j<k<l<p<q<r<s}\left|\mathbb{E}X_iX_jX_kX_lX_pX_qX_rX_s\right|
\end{aligned}$$

Fix $j - i = r$. There are $(n-r)$ choices (at most) for the pair $(i,\, j)$. Subsequently, there are (at most) $(n-r)^6$ choices for the remaining indices. Hence, by Doukhan (1994), Section 1.2.2, Theorem 3, we have that, for some universal constant $C \in \mathbb{R}_+$,

$$\begin{aligned}
\frac{C}{n^8}\sum_{i<j<k<l<p<q<r<s}\left|\mathbb{E}X_iX_jX_kX_lX_pX_qX_rX_s\right| &\leq \frac{C}{n}\sum_{r=1}^{n-1}\left(\frac{n-r}{n}\right)^7\alpha_X(r)^{\frac{\delta}{2+\delta}} \\
&= O\left(\frac{1}{n}\right)\ .
\end{aligned}$$

The result follows. ■

PROOF OF LEMMA 3.3. By an identical argument to the one used in the proof of Lemma S.3.4, we have that

$$\hat{K}_n^2 \xrightarrow{p} \text{Var}\left(X_1^2\right)$$
$$\hat{\nu}_1 \xrightarrow{p} 0 \ .$$

Assuming, without loss of generality, that $\mathbb{E}X_1 = 0$, we have that, as a consequence of Theorem 3.2, Hoeffding's condition (see Chung and Romano (2013), Theorem 5.1), and Slutsky's Theorem, the sample autocorrelation $\hat{\rho}_n$ under random permutation satisfies

$$\hat{\rho}_n \xrightarrow{p} 0 \ .$$

Hence, by Slutsky's theorem and Lemma S.3.4, it follows that

$$\hat{\gamma}_n^2 \xrightarrow{p} 1 \ ,$$

as required. ∎

Using the previous results, we may now prove Theorem 3.3.

Proof of Theorem 3.3. By Theorem 3.2, Lemma 3.3, and Slutsky's theorem for randomization distributions (Chung and Romano (2013)), (3.23) holds.

By Theorem 2.3, Lemma 3.2, and Slutsky's theorem, (3.22) holds. ∎

We illustrate the proof of Theorem 3.4, which proceeds almost identically to the proof of Theorem 3.1.

Proof of Theorem 3.4. The proof is exactly the same as that of Theorem 3.1, since we have that, for each fixed r, we have that there exists a constant $C \in \mathbb{R}_+$ such that

$$\sum_{i=1}^{r-1} \alpha_{X^{(r)}}(i) < C \ ,$$

and the result of Wald and Wolfowitz (1943) still applies. The only difference is that the definition of the random variable $B(m)$ (see S.37) becomes

$$B(m) = \sum_{i=1} \frac{1}{m} \sum_{i=1}^{m} \left(X_i^{(m)}\right)^2 - \frac{1}{m^2} \left(\sum_{j=1}^{m} X_j^{(m)}\right)^2 \ .$$

The remainder of the proof is identical. ∎

S.4 Testing multiple lags

Proof of Theorem 4.1. We may assume, without loss of generality, that For $i \in \mathbb{N}$, and $\{a_i,\, i \in \{0,\, 1,\, \dots,\, k\}\} \subset \mathbb{R}^{k+1}$, let

$$Y_i = \sum_{k=0}^{r} a_i \left(X_i - \bar{X}_n\right)\left(X_{i+k} - \bar{X}_n\right) \ .$$

Let

$$Z_i = \sum_{k=0}^{r} a_i X_i X_{i+k} \ .$$

By the same argument as in the proof of Theorem 2.3, we have that

$$Y_i = Z_i + o_p(\sqrt{n}) \ .$$

Also, note that the sequence (Z_i) is α-mixing, with α-mixing coefficients given by

$$\alpha_Z(i) = \alpha_X(i - r) \ .$$

Noting also that

$$E[Z_i] = a_0 \sigma^2 + \sum_{k=1}^{r} a_k \rho_k \sigma^2 \ ,$$

by applying Ibragimov's central limit theorem as in the proof of Theorem 2.3, we may conclude that

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n-1} Y_i - \sqrt{n}\left(a_0\sigma^2 + \sum_{k=1}^{r} a_k \rho_k \sigma^2\right) \xrightarrow{d} N\left(0,\, \tilde{\sigma}^2\right) \ ,$$

where

$$\begin{aligned}\tilde{\sigma}^2 &= \mathrm{Var}(Y_1) + 2\sum_{k>1} \mathrm{Cov}(Y_1,\, Y_k) \\ &= \sum_{k=0}^{r} a_k^2 \cdot \mathrm{Var}(X_1 X_{1+k}) + 2\sum_{j<k} a_j a_k \mathrm{Cov}(X_1 X_{1+j},\, X_1 X_{1+k}) + 2\sum_{l>1}\sum_{j=0}^{r}\sum_{k=0}^{r} a_j a_k \mathrm{Cov}(X_1 X_{1+j},\, X_l X_{l+k}) \ .\end{aligned}$$

Since the a_i were arbitrary, it follows that, by the Cramér-Wold device,

$$\sqrt{n}\left[\hat{\sigma}_n^2\begin{pmatrix}1\\ \hat{\rho}_1\\ \vdots\\ \hat{\rho}_r\end{pmatrix}-\sigma^2\begin{pmatrix}1\\ \rho_1\\ \vdots\\ \rho_r\end{pmatrix}\right]\xrightarrow{d} N(0,\,\Sigma)\ .$$

We may now apply the delta method, with the function $h(x_0,\,\ldots,\,x_r)=(x_1/x_0,\,x_2/x_0,\,\ldots,\,x_r/x_0)$, to conclude that

$$\sqrt{n}\left[\begin{pmatrix}\hat{\rho}_1\\ \vdots\\ \hat{\rho}_r\end{pmatrix}-\begin{pmatrix}\rho_1\\ \vdots\\ \rho_r\end{pmatrix}\right]\xrightarrow{d} N\left(0,\,A^T\Sigma A\right)\ ,$$

where

$$A=\begin{pmatrix}-\frac{\rho_1}{\sigma^4} & \ldots & \ldots & \ldots & -\frac{\rho_r}{\sigma^4}\\ \frac{1}{\sigma^2} & 0 & \ldots & \ldots & 0\\ 0 & \frac{1}{\sigma^2} & 0 & \ldots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & \ldots & \ldots & \ldots & \frac{1}{\sigma^2}\end{pmatrix}\ ,$$

as required. ■

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