Stanford University  
Departments of Mathematics and Statistics  

PROBABILITY SEMINAR  
4pm, Monday, April 29, 2019  
Sequoia Hall Room 200  
Refreshments served at 3:30pm in the Lounge.

Speaker: Jim Pitman  
Departments of Statistics and Mathematics,  
University of California, Berkeley

Title: Probability laws with a power biased inverse: Functional and distributional equations related to self-reciprocal Fourier transforms and the Riemann zeta function

Abstract:  
The functional equation \( f(1/t) = t^{2+q} f(t) \) for a probability density function \( f \) on the positive half line, and some real \( q \), is of interest in several contexts. It is known that \( f \) solves this equation with \( q = -n \) iff for a random vector \( Z_n \) with \( n \) independent standard Gaussian components, independent of \( \Sigma \) with density \( f \), the probability density of \( \Sigma Z_n / \sqrt{2\pi} \) on \( R^n \) is self-reciprocal in the sense of harmonic analysis, meaning it is its own Fourier transform. For a general distribution of \( \Sigma \), the density of \( \Sigma Z_n / \sqrt{2\pi} \) is self-reciprocal iff \( Y = \Sigma^{-n} \) has a size-biased inverse, meaning \( EYg(Y) = Eg(Y^{-1}) \) for all nonnegative measurable \( g \). Equivalently, \( EY^{1-s} = EY^s \) for all \( 0 \leq s \leq 1 \). This is a distributional symmetry of \( Y \) in the same vein as the Palm–Mecke–Chen characterization of the Poisson distribution, and Stein’s characterization of the normal distribution, by identities involving \( EYg(Y) \) for a generic function \( g \). Biane and Yor showed a distribution of \( Y \) with a size-biased inverse is involved in the probabilistic representation of Riemann’s \( \xi \) function by \( 2\xi(s) = EY^s \). Then the distributional symmetry of \( Y \) is equivalent to Riemann’s functional equation \( \xi(1-s) = \xi(s) \), leading to the functional equation and analytic continuation of the Riemann zeta function. The set of all distributions of \( Y \) with a size-biased inverse may be characterized in a number of ways. In particular, it is a simplex: every such distribution is a unique probabilistic mixture over \( 0 < u \leq 1 \) of the extreme distribution of \( Y_u \) on \( \{u, u^{-1}\} \) with mean 1.