CALCULATION OF UPPER TAIL PERCENTILES FOR THE CHI-SQUARE DISTRIBUTION

BY

HERMAN RUBIN and JAMES V. ZIDEK

TECHNICAL REPORT NO. 102

DECEMBER 1, 1964

THIS RESEARCH WAS SPONSORED BY THE ARMY RESEARCH OFFICE, OFFICE OF NAVAL RESEARCH, AND AIR FORCE OFFICE OF SCIENTIFIC RESEARCH BY CONTRACT NO. Nonr-225(52) (NR 342-022)

DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA

CALCULATION OF UPPER TAIL PERCENTILES FOR THE CHI-SQUARE DISTRIBUTION

by

Herman Rubin and James V. Zidek

TECHNICAL REPORT No. 102

December 1, 1964

PREPARED UNDER CONTRACT Nonr-225(52)

(NR-342-022)

FOR

OFFICE OF NAVAL RESEARCH

DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA

CALCULATION OF UPPER TAIL PERCENTILES FOR THE

CHI-SQUARE DISTRIBUTION

by

Herman Rubin and James V. Zidek

1. Summary

This report describes a procedure for determining upper tail percentiles of the chi-square distribution with an arbitrary number of degrees of freedom. It consists of two main parts. The first is a description of an effective iterative technique, which is to be published by H. Rubin, for solving equations of the form $f(x) = 0$, given the feasibility of computing f and its derivatives. The second, in the application of this technique to the chi-square distribution, is the continued fraction approximation to the tail integral of the chi-square density function. An alternative procedure is given for the cases in which the sequence of approximants to this continued fraction converge slowly.

2. Iterative Solution of $f(x) = 0$.

In general, let $f(x)$ denote a real valued function defined on the real line. Suppose f can be represented as a power series, that is,

$$f(x+y) = \sum_{n=0}^{\infty} a_n(x)y^n , \tag{1}$$

where $a_n(x) = f^{(n)}(x)/n!$

In the problem with which this report is concerned, we are interested in solving equations of the form,

(2) $$f(x) = 0 .$$

Suppose x_o is an initial approximation to the required solution. If x' denotes this solution, let

$$x'-x_o = (t+\gamma t^2)/(1-\alpha t-\beta t^2) ,$$

where α, β, and γ are constants chosen in accordance with the following considerations.

Using equation (1), we have

(3) $$(1-\alpha t-\beta t^2)f(x_o + \frac{t+\gamma t^2}{1-\alpha t-\beta t^2}) = \sum_{n=0}^{\infty} a_n(x_o) \frac{(t+\gamma t^2)^n}{(1-\alpha t-\beta t^2)^{n-1}} .$$

If α, β, and γ are chosen so that (α,β,γ) is one of the solutions of the equations

(4) $$\begin{gathered} 2\gamma+\alpha+a_3(x_o)/a_2(x_o) = 0 \\ \alpha^2+\gamma^2+2\alpha\gamma+\beta+3\gamma a_3(x_o)/a_2(x_o)+ \\ 2\alpha a_3(x_o)/a_2(x_o)+a_4(x_o)/a_2(x_o) = 0 , \end{gathered}$$

it is easily shown that the coefficients of t^3 and t^4 on the right hand side of equation (3) vanish while that of t^5 becomes $2a_3^3(x_o)/a_2^2(x_o)-3a_3(x_o)a_4(x_o)/a_2(x_o)+a_5(x_o)$. Let $(\alpha_o,\beta_o,\gamma_o)$ denote a solution of equations (4). Then an approximation, t_o, to the value of t for which

$$x'-x_o = (t+\gamma_o t^2)/(1-\alpha_o t-\beta_o t^2),$$

is the smaller solution, in absolute value, of

$$(1-\alpha_o t-\beta_o t^2)f(x_o) + f^{(1)}(x_o)(t+\gamma_o t^2) + \frac{1}{2} f^{(2)}(x_o)t^2 = 0 ,$$

that is,

$$t_o = 2f(x_o)[\alpha_o f(x_o)-f^{(1)}(x_o)\pm\{(\alpha_o^2+4\beta_o)f^2(x_o)-2(2\gamma_o+\alpha_o)f^{(1)}(x_o)f(x_o)+(f^{(1)}(x_o))^2-2f(x_o)f^{(2)}(x_o)\}^{\frac{1}{2}}]^{-1},$$

where the plus or minus sign is chosen according as $\alpha_o f(x_o)-f^{(1)}(x_o)$ is positive or negative.

Let $x_1 = x_o+(t_o+\gamma_o t_o^2)/(1-\alpha_o t_o-\beta_o t_o^2)$. Then x_1 is an improved approximation of x'. Define a sequence, $\{x_i\}$, of successive approximations of x', by the relation

$$x_n = x_{n-1}+(t_{n-1}+\gamma_{n-1}t_{n-1}^2)/(1-\alpha_{n-1}t_{n-1}-\beta_{n-1}t_{n-1}^2)\,(n = 1,2,\dots)$$

where $(\alpha_{n-1},\beta_{n-1},\gamma_{n-1})$ is one of the solutions of equations (4) and

$$(5)\quad t_{n-1} = 2f(x_{n-1})[\alpha_{n-1}f(x_{n-1})-f^{(1)}(x_{n-1})\pm\{(\alpha_{n-1}f(x_{n-1})-f^{(1)}(x_{n-1}))^2-4f(x_{n-1})(\gamma_{n-1}f^{(1)}(x_{n-1})-\beta_{n-1}f(x_{n-1})+\tfrac{1}{2}f^{(2)}(x_{n-1}))\}^{\frac{1}{2}}]^{-1}$$

the plus or minus sign being chosen according as $\alpha_{n-1}f(x_{n-1})-f^{(1)}(x_{n-1})$ is positive or negative.

3. Computation of Percentiles of the Chi-Squared Distribution (p small).

The procedure just described can be applied to the problem of computing upper tail percentiles for the chi-squared distribution with $k(k=1,2,\dots)$ degrees of freedom. Let $\chi^2(k)$ denote the random variable having a chi-squared distribution with k degrees of freedom, and $\chi_p^2(k)$, the $(1-p)^{th}$ percentile for this distribution. Then $\chi^2(k)$

has probability density function

$$g_{\chi^2(k)}(x)dx = 2^{-k/2}\Gamma^{-1}(k/2)x^{k/2-1}e^{-x/2}dx \qquad (x > 0)$$
$$= 0 \qquad (x \leq 0) .$$

If we let $y(k) = \chi^2(k)/2$, $y(k)$ has probability density function

$$g_{y(k)}(x)dx = \Gamma^{-1}(k/2)x^{k/2-1}e^{-x}dx \qquad (x > 0)$$
$$= 0 \qquad (x \leq 0) ,$$

that is, $y(k)$ has the gamma distribution with parameter $k/2$. If $y_p(k)$ denotes the $(1-p)^{th}$ percentile for this distribution,

(6) $$\chi_p^2(k) = 2y_p(k) .$$

For simplicity, we determine $y_p(k)$ using the procedure described in section 2, and $\chi_p^2(k)$ is then obtained using equation (6).

To this end, let

$$f(x) = \log_e\left\{\frac{1}{\Gamma(\frac{k}{2})}\int_x^\infty e^{-t}t^{k/2-1}dt\right\} - \log_e p, \quad (0 < p < 1) .$$

Then, if $g(x)$ and $h(x)$ denote $-e^{-x}x^{\frac{k}{2}-1}$ and $\int_x^\infty e^{-t}t^{\frac{k}{2}-1}dt$, respectively,

(7)
$$f^{(1)} = g/h$$
$$f^{(2)} = h^{-1}[g^{(1)}-g(\tfrac{g}{h})]$$
$$f^{(3)} = h^{-1}[g^{(2)}-3g^{(1)}(\tfrac{g}{h})+2g(\tfrac{g}{h})^2]$$
$$f^{(4)} = h^{-1}[g^{(3)}-4g^{(2)}(\tfrac{g}{h})-3(g^{(1)})^2/h+12g^{(1)}(\tfrac{g}{h})^2-6g(\tfrac{g}{h})^3] .$$

Note that in the computation of these derivatives significant figures will be lost through cancellation.

Now, using the even part of expansion (92.9) given by Wall [1, page 356],

$$(8) \qquad h(x) = \cfrac{e^{-x}x^{\frac{k}{2}}}{x+1-\frac{k}{2}+\cfrac{(\frac{k}{2}-1)\cdot 1}{x+3-\frac{k}{2}+\cfrac{(\frac{k}{2}-2)\cdot 2}{x+5-\frac{k}{2}+\ddots}}} \; .$$

Table 1 in the appendix is comprised of calculations which indicate the rate at which the sequence of successive approximants to $h(x)$ converges. When p is moderate and k is large, this convergence is slow. Therefore, it is preferable, in these cases, to use the alternate procedure, for the calculation of $\chi_p^2(k)$, described in Section 4 of this report.

At the same time,

$$g^{(1)}(x) = -g(x)\left(\frac{x-\frac{k}{2}+1}{x}\right)$$

$$(9) \qquad g^{(2)}(x) = g(x)\left[\left(\frac{x-\frac{k}{2}+1}{x}\right)^2 - \frac{(\frac{k}{2}-1)}{x^2}\right]$$

$$g^{(3)}(x) = -g(x)\left[\left(\frac{x-\frac{k}{2}+1}{x}\right)^3 - \frac{3(\frac{k}{2}-1)}{x^2}\left(\frac{x-\frac{k}{2}+1}{x}\right) - \frac{2(\frac{k}{2}-1)}{x^3}\right] .$$

For the initial estimate, x_o, we take, using the Hilferty-Wilson approximation (cf., Kendall and Stuart, [2], page 373)

(10) $$x_o = k[2^{\frac{2}{3}}(\frac{1}{2} - \frac{1}{9k}) + z_p(\frac{2^{\frac{1}{3}}}{9k})^{\frac{1}{2}}]^3 ,$$

where z_p is defined by

$$\frac{1}{\sqrt{2\pi}} \int_{z_p}^{\infty} e^{-t^2/2} dt = p ,$$

and may be obtained, with a high degree of accuracy, using, for example, tables provided by the National Bureau of Standards [3].

Kendall and Stuart ([2], page 374) give a table containing calculated values of x_o for several values of p and k. For example, they give for $k = 42$, the values 29.060 and 33.113 for $p = 0.05$ and $p = 0.01$, while for $k = 82$, 52.068 and 57.355 are given for $p = 0.05$ and $p = 0.01$. These compare favorably with the correct values of 29.063, 33.103, 52.069, and 57.347, respectively.

A summary of the procedure is given below. It should be noted that by choosing

$$\alpha_n = 0$$

(11) $$\gamma_n = -\frac{1}{6} f^{(3)}(x_n)/f^{(2)}(x_n)$$

$$\beta_n = -\frac{1}{12} f^{(4)}(x_n)/f^{(2)}(x_n) + \frac{5}{36} [f^{(3)}(x_n)/f^{(2)}(x_n)]^2 ,$$

$(n=0,1,2,\ldots)$, some computational simplifications occur.

With p and k specified, find z_p and x_o using equation (10). Having determined $x_{n-1} (n=1,2,\ldots)$, compute $f(x_{n-1})$ and $f^{(i)}(x_{n-1})$ $(i=1,2,3,4)$ using representation (8) and equations (7), respectively. Calculate $\alpha_{n-1}, \beta_{n-1}$, and γ_{n-1}. Then compute t_{n-1} using equation (5).

If the values suggested in equations (11) are used, we have

$$(12) \qquad t_{n-1} = -2f(x_{n-1})[f^{(1)}(x_{n-1})-\{4(\beta_{n-1}f^2(x_{n-1}) -\gamma_{n-1}f^{(1)}(x_{n-1})f(x_{n-1})) + (f^{(1)}(x_{n-1}))^2 -2f(x_{n-1})f^{(2)}(x_{n-1})\}^{\frac{1}{2}}]^{-1} .$$

Finally,

$$x_n = x_{n-1} + (t_{n-1}+\gamma_{n-1}t^2_{n-1})/(1-\alpha_{n-1}t_{n-1}-\beta_{n-1}t^2_{n-1}) .$$

In this way, a sequence, $\{x_i\}$, of successive approximations is obtained. The calculations are complete when the prescribed degree of accuracy is attained.

Now, if the last member of this sequence is denoted by $y_p(k)$, the $(1-p)^{th}$ percentile for the chi-square distribution with k degrees of freedom is, to the prescribed degree of accuracy,

$$\chi^2_p(k) = 2y_p(k) .$$

This procedure, using the values suggested in equation (11) is illustrated in Table 2.

4. Alternate Method for Computation of $\chi_p(k)$ (medium p, large k).

As was pointed out in Section 3 it is difficult to apply the iterative method of that section because the continued fraction approximation to the integral converges slowly when p is moderate and k is large. In this section, we describe an alternate method suitable for the calculation of $\chi^2_p(k)$ in these cases.

Define a random variable, T_k by

$$(\nu^{1/3} + \frac{1}{3} T_k \nu^{-1/6})^3 = y(k) ,$$

where $\nu = k/2-1/3$ and $y(k)$ denotes the random variable having the gamma distribution with parameter $k/2$. Then T_k has density

$$(13) \qquad f_T(t) = \frac{\nu^{-1/6}}{\Gamma(\nu+1/3)} e^{-(\nu^{1/3}+ \frac{1}{3} t \nu^{-1/6})^3} (\nu^{1/3}+ \frac{1}{3} t\nu^{-1/6})^{3\nu} \qquad (0 < t < \infty),$$

where, for convenience, the subscript, k on T_k has been suppressed. Using the asymptotic expansion (cf. Whittaker and Watson, [4], page 252),

$$\log \Gamma(x) = (x - \frac{1}{2})\log x - x + \frac{1}{2} \log(2\pi) + \frac{1}{12x} - \frac{1}{360x^3} + \cdots ,$$

we obtain

$$f_T(t) \sim \phi(t)(1 + \frac{1}{36\nu} + \frac{17}{2592\nu^2} + \cdots)(1 - \frac{t^4}{108\nu} + \frac{t^5}{405\nu^{3/2}} + \frac{1}{\nu^2} [\frac{t^8}{23328} - \frac{t^6}{1458}] + \frac{1}{\nu^{5/2}} [\frac{t^7}{5103} - \frac{t^9}{43740}] + \cdots) ,$$

where $\phi(t) = e^{-t^2/2}/\sqrt{2\pi}$. Thus

$$(14) \qquad \int_\xi^\infty f_T(t)dt \sim N_o(\xi) + \frac{1}{\nu}[\frac{1}{36} N_o(\xi) - \frac{1}{108} N_4(\xi)] + \frac{1}{\nu^{3/2}} [\frac{N_5(\xi)}{405}] + \frac{1}{\nu^2} [\frac{17 N_o(\xi)}{2592} - \frac{N_4(\xi)}{3888} - \frac{N_6(\xi)}{1458} + \frac{N_8(\xi)}{23328}] + \frac{1}{\nu^{5/2}} [\frac{N_5(\xi)}{14580} + \frac{N_7(\xi)}{5103} - \frac{N_9(\xi)}{43740}] + \cdots ,$$

where

$$N_r(\xi) = \frac{1}{\sqrt{2\pi}} \int_\xi^\infty e^{-\frac{t^2}{2}} t^r dt \qquad (r=1,2,\ldots) .$$

If we denote the coefficient of $(\nu^{-1/2})^j$ $(j=2,3,\ldots)$, in expansion (14), by $G_j(\xi)$ and expand $\int_\xi^\infty f_T(t)dt$ about z_p, we obtain,

$$(15) \qquad \int_\xi^\infty f_T(t)dt = p + \sum_{n=2}^\infty \nu^{-n/2} G_n(z_p) + (N_o'(z_p) + \sum_{n=2}^\infty \nu^{-n/2} G_n'(z_p))$$

$$\times (\xi - z_p) + \tfrac{1}{2}(N_o''(z_p) + \sum_{n=2}^\infty \nu^{-n/2} G_n''(z_p))(\xi - z_p)^2 + \cdots .$$

Let

$$R(z_p, \nu) = z_p + \sum_{n=2}^\infty \nu^{-n/2} c_n(z_p)$$

where the $\{c_i\}_{i=2}^\infty$ are chosen so that

$$\int_{R(z_p,\nu)}^\infty f_T(t)dt = p$$

for all ν, that is, to make $R(z_p,\nu)$ the $(1-p)^{th}$ percentile for the random variable whose density is given by equation (13). It is easily verified that

$$c_2 = G_2(z_p)/\phi(z_p) , \quad c_3 = G_3(z_p)/\phi(z_p) ,$$

$$c_4 = \phi^{-1}(z_p)[G_4(z_p) + c_2 G_2'(z_p) + \tfrac{1}{2} c_2^2 \phi(z_p) z_p] ,$$

$$c_5 = \phi^{-1}(z_p)[G_5(z_p) + c_3 G_2'(z_p) + c_2 G_3'(z_p) + c_2 c_3 z_p \phi(z_p)]$$

Upon application of the relations, $N_r(z) = \phi(z) z^{r-1} + (r-1) N_{r-2}(z)$ and

$N_r'(z) = -z^r \phi(z) \quad (r=1,2,\ldots)$, and simplification we obtain

$$c_2 = -\frac{1}{108}(z_p^3 + 3z_p) \quad c_3 = \frac{1}{405}(z_p^4 + 4z_p^2 + 8)$$

$$c_4 = -\frac{1}{2592}(z_p^5 + 4z_p^3 + 15z_p)$$

$$c_5 = \frac{1}{76545}(z_p^6 - 15z_p^4 - 18z_p^2 + 48).$$

Finally, the required percentile for the chi-square distribution with k degrees of freedom is

$$\chi_p^2(k) = 2(\nu^{1/3} + \frac{1}{3} R(z_p,\nu)\nu^{-1/6})^3 ,$$

using equation (6) and since

$$y_p(k) = (\nu^{1/3} + \frac{1}{3} R(z_p,\nu)\nu^{-1/6})^3 .$$

Calculations, using the approximation,

$$R(z_p,\nu) \approx z_p + c_2\nu^{-1} + c_3\nu^{-3/2} + c_4\nu^{-2} + c_5\nu^{-5/2}$$

yielded, in the cases $k = 30$ and $P = 0.1, 0.01, 0.001$, the tabulated values for χ^2, of 40.2560, 50.892, and 59.703, respectively.

APPENDIX

Table 1.

Successive approximations,

$$D_n = \frac{e^{-x}x^{k/2}}{x+1-k/2+}\;\cdot\cdot\cdot + \frac{[k/2-(n-1)](n-1)}{x+(2n-1)-k/2}\;,$$

to $h(x)$ for various values of x and several successive integers n and $k = 5$.

	n	D_n
	1	0.1116631751 (-15)
	2	0.1105992097 (-15)
x=40	3	0.1105998181 (-15)
	4	0.1105998186 (-15)
	5	0.1105998186 (-15)
	1	0.168902569 (-2)
	2	0.166110791 (-2)
	3	0.1661315547 (-2)
x=10	4	0.1661317266 (-2)
	5	0.1661317311 (-2)
	6	0.1661317313 (-2)
	7	0.1661317313 (-2)
	1	0.1076179107
	2	0.0998383026
	3	0.1000094174
	4	0.1000130146
x=5	5	0.1000132282
	6	0.1000132401
	7	0.1000132506
	8	0.1000132509

Table 2.

Calculations illustrating the use of the procedure described in Section 3. Here $k = 6$, $p = 10^{-15}$, $z_p \approx 7.94$ and $x_o = 46.4$.

$f(x_o) = -4.83668\ 06556\ 61$	$\alpha_o = 0$
$f^{(1)}(x_o) = -0.95782\ 46788$	$\gamma_o = 0.00702\ 29602\ 0$
$f^{(2)}(x_o) = -0.00088\ 89830\ 100$	$\beta_o = 0.00002\ 47601\ 2325$
$f^{(3)}(x_o) = 0.00003\ 74597\ 5383$	$t_o = -5.15496\ 3915$
$f^{(4)}(x_o) = -0.00000\ 23666\ 47654$	$x_1 = 41.42839\ 064$
$f(x_1) = -0.08657\ 06189\ 89$	$\alpha_1 = 0$
$f^{(1)}(x_1) = -0.95288\ 79229$	$\gamma_1 = 0.00784\ 32843\ 50$
$f^{(2)}(x_1) = -0.00110\ 91574\ 19$	$\beta_1 = 0.00003\ 09130\ 1835$
$f^{(3)}(x_1) = 0.00005\ 21966\ 2219$	$t_1 = -0.09092\ 02279$
$f^{(4)}(x_1) = -0.00000\ 36824\ 80654$	$x_2 = 41.33753\ 504$
$f(x_2) = 6.1 \times 10^{-9}$ *	

* At this stage, the error, $x'-x_3$, is approximately $f(x_2)$. Thus

$$\chi_p^2(6) \approx 82.67507008,$$

with $p = 10^{-15}$.

REFERENCES

[1] Wall, H.S., Analytic Theory of Continued Fractions. New York: D. Van Nostrand Co., Inc., 1948.

[2] Kendall, M.G. and Stuart, A., The Advanced Theory of Statistics. Volume I. London: C. Griffin and Co., 1958.

[3] National Bureau of Standards, Tables of Probability Functions, Volume II, 1942.

[4] Whittaker, E.T. and Watson, G.N., A Course of Modern Analysis, Fourth Edition, Cambridge: Cambridge University Press, 1963.

Unclassified

Security Classification

DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)* Stanford University Department of Statistics Stanford, California	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE

Calculations of Upper Tail Percentiles for the Chi-Square Distribution

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*
Technical Report, December 1, 1964

5. AUTHOR(S) *(Last name, first name, initial)*

Rubin, Herman and Zidek, James V.

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
December 1, 1964	Thirteen	Five

8a. CONTRACT OR GRANT NO. Nonr 225 (52) b. PROJECT NO. NR 342-022 c. Nonr 225(52) d.	9a. ORIGINATOR'S REPORT NUMBER(S) Technical Report No. 102 9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)*

10. AVAILABILITY/LIMITATION NOTICES

Releaseable without limitations on dissemination

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY Logistics and Mathematical Sciences Branch-Office of Naval Research Washington, D.C. 20360

13. ABSTRACT

This report describes a procedure for determining upper tail percentiles of the chi-square distribution with an arbitrary number of degrees of freedom. It consists of two main parts. The first is a description of an effective iterative technique, which is to be published by H. Rubin, for solving equations of the form $f(x) = 0$, given the feasibility of computing f and its derivatives. The second, in the application of this technique to the chi-square distribution, is the continued fraction approximation to the tail integral of the chi-square density function. An alternative procedure is given for the cases in which the sequence of approximants to this continued fraction converge slowly.

DD FORM 1 JAN 64 1473

Unclassified

Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Mathematical Models of Group Behavior Group Problem-Solving Individuals vs. Group Problem-Solving						

INSTRUCTIONS

1. ORIGINATING ACTIVITY: Enter the name and address of the contractor, subcontractor, grantee, Department of Defense activity or other organization *(corporate author)* issuing the report.

2*a*. REPORT SECURITY CLASSIFICATION: Enter the overall security classification of the report. Indicate whether "Restricted Data" is included. Marking is to be in accordance with appropriate security regulations.

2*b*. GROUP: Automatic downgrading is specified in DoD Directive 5200.10 and Armed Forces Industrial Manual. Enter the group number. Also, when applicable, show that optional markings have been used for Group 3 and Group 4 as authorized.

3. REPORT TITLE: Enter the complete report title in all capital letters. Titles in all cases should be unclassified. If a meaningful title cannot be selected without classification, show title classification in all capitals in parenthesis immediately following the title.

4. DESCRIPTIVE NOTES: If appropriate, enter the type of report, e.g., interim, progress, summary, annual, or final. Give the inclusive dates when a specific reporting period is covered.

5. AUTHOR(S): Enter the name(s) of author(s) as shown on or in the report. Enter last name, first name, middle initial. If military, show rank and branch of service. The name of the principal author is an absolute minimum requirement.

6. REPORT DATE: Enter the date of the report as day, month, year; or month, year. If more than one date appears on the report, use date of publication.

7*a*. TOTAL NUMBER OF PAGES: The total page count should follow normal pagination procedures, i.e., enter the number of pages containing information.

7*b*. NUMBER OF REFERENCES: Enter the total number of references cited in the report.

8*a*. CONTRACT OR GRANT NUMBER: If appropriate, enter the applicable number of the contract or grant under which the report was written.

8*b*, 8*c*, & 8*d*. PROJECT NUMBER: Enter the appropriate military department identification, such as project number, subproject number, system numbers, task number, etc.

9*a*. ORIGINATOR'S REPORT NUMBER(S): Enter the official report number by which the document will be identified and controlled by the originating activity. This number must be unique to this report.

9*b*. OTHER REPORT NUMBER(S): If the report has been assigned any other report numbers *(either by the originator or by the sponsor)*, also enter this number(s).

10. AVAILABILITY/LIMITATION NOTICES: Enter any limitations on further dissemination of the report, other than those imposed by security classification, using standard statements such as:

(1) "Qualified requesters may obtain copies of this report from DDC."

(2) "Foreign announcement and dissemination of this report by DDC is not authorized."

(3) "U. S. Government agencies may obtain copies of this report directly from DDC. Other qualified DDC users shall request through ________________."

(4) "U. S. military agencies may obtain copies of this report directly from DDC. Other qualified users shall request through ________________."

(5) "All distribution of this report is controlled. Qualified DDC users shall request through ________________."

If the report has been furnished to the Office of Technical Services, Department of Commerce, for sale to the public, indicate this fact and enter the price, if known.

11. SUPPLEMENTARY NOTES: Use for additional explanatory notes.

12. SPONSORING MILITARY ACTIVITY: Enter the name of the departmental project office or laboratory sponsoring *(paying for)* the research and development. Include address.

13. ABSTRACT: Enter an abstract giving a brief and factual summary of the document indicative of the report, even though it may also appear elsewhere in the body of the technical report. If additional space is required, a continuation sheet shall be attached.

It is highly desirable that the abstract of classified reports be unclassified. Each paragraph of the abstract shall end with an indication of the military security classification of the information in the paragraph, represented as *(TS), (S), (C), or (U)*.

There is no limitation on the length of the abstract. However, the suggested length is from 150 to 225 words.

14. KEY WORDS: Key words are technically meaningful terms or short phrases that characterize a report and may be used as index entries for cataloging the report. Key words must be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location, may be used as key words but will be followed by an indication of technical context. The assignment of links, roles, and weights is optional.