ON A CHARACTERIZATION OF THE THREE LIMITING TYPES OF THE EXTREME

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In the recent papers, Rao (1964), and Rao and Rubin (1964), some special damage models associated with discrete distributions have been discussed and a characterization of the Poisson distribution has been obtained in the latter. In this note we describe a damage model that arises rather frequently among continuous distributions and obtain a characterization of the three limiting types of the extreme in terms of this model.

Let X be a random variable with a continuous distribution function. The model we wish to describe is the following: when the statistician starts to measure X, an independent random variable Y with a continuous distribution function influences the experiment and he is forced to observe a new random variable Z which is $\min(X,Y)$. We assume that the statistician can also examine his experiment and determine whether $Z = X$ (that is, the observation is an undamaged one) or whether $Z = Y$ (that is, the observation is a damaged one). As an illustration, consider the experiment of determining the life of an electrical bulb. What one observes when the bulb stops burning is actually $Z = \min(X,Y)$ where X is the life of the bulb and Y is the life of the electric circuit, consisting of fuses,

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batteries etc., feeding electricity to the bulb. On further inspection one can determine whether the bulb fused or whether the circuit broke down, that is, whether $Z = X$ or $Z = Y$. The case $Z = X = Y$ is excluded by the assumption that X and Y have continuous distribution functions.

In this note it is to be understood that all random variables are independently distributed and have continuous distribution functions.

Definition: X and Y are said to be compatible if $\text{Prob}(Y > X) > 0$ and $\text{Prob}(X > Y) > 0$.

Definition: The damage of Y on X is confounding if the distribution of Z and the conditional distribution of Z given $Z = X$ are the same.

In other words the damage of Y on X is confounding if taking care to observe only the undamaged observations leads to the same results as observing all the observations. We also note that the concept of the damage of Y on X being confounding becomes meaningful only when Y and X are compatible.

Let the distribution function of X, of Y, of $Z = \min(X,Y)$, the conditional distribution function of X given $Z = X$ and of Z given $Z = Y$, be $F(x)$, $G(x)$, $H_1(x)$, $H_2(x)$, and $H_3(x)$ respectively. Also let X and Y be compatible. Then

$$
\begin{aligned}
dH_1(x) &= (1-G(x))dF(x) + (1-F(x))dG(x) \\
(1) \qquad dH_2(x) &= [(1-G(x))dF(x)]/[\int(1-G(y))dF(y)] \\
dH_3(x) &= [(1-F(x))dG(x)]/[\int(1-F(y))dG(y)]
\end{aligned}
$$

Remark 1: The damage of Y on X is confounding if $F(x) \equiv G(x)$.

Remark 2: When Y and X are compatible the damage of Y on X is confounding if and only if the damage of X on Y is confounding.

From (1) we have

$$(2)\qquad [\int(1-G(y))dF(y)]dH_2(x) + [\int(1-F(y))dG(y)]dH_3(x) = dH_1(x) .$$

Again,

$$[\int(1-G(y))dF(y)] + [\int(1-F(y))dG(y)] = 1 ,$$

and $[\int(1-G(y))dF(y)] > 0$ and $[\int(1-F(y))dG(y)] > 0$ since X and Y are compatible. Thus $H_1(x) = H_2(x)$ if and only if $H_1(x) = H_3(x)$.

Theorem 1. When X and Y are compatible the damage of Y on X is confounding if and only if there is a $p > o$ such that

$$(3)\qquad (1-G(x))^p = (1-F(x)) .$$

Proof: Let the damage of Y on X be confounding. From remark 2, $H_2(x) = H_3(x)$. Thus $(1-G(x))dF(x) = p(1-F(x))dG(x)$ where $p > o$. Thus

$$\log(1-F(x)) = p \log(1-G(x)) + \log c$$

or

$$c(1-G(x))^p = (1-F(x)) .$$

Since F and G are distribution functions c must be equal to 1. Hence the 'only if' part.

Conversely, let (3) hold. Then by direct calculation

$$dH_1(x) = dH_2(x) = dH_3(x) = (p+1)(1-G(x))^p dG(x) .$$

Hence the theorem.

Remark 3. The above theorem also shows that if the damage of Y on X is confounding and X and Y are compatible then the range of X is the same as the range of Y.

Remark 4. Let Y_1 and X be compatible and Y_2 and X be compatible. If the damage of Y_1 on X is confounding and the damage of Y_2 on X is confounding then the damage of Y_1 on Y_2 is confounding. Let the distribution functions of X, Y_1 and Y_2 be $F(x)$, $G_1(x)$ and $G_2(x)$, respectively. Then from Theorem 1, there exist $p_1 > o$, $p_2 > o$ such that

$$(1-G_i(x))^{p_i} = (1-F(x)) \qquad i=1,2 .$$

Thus $(1-G_2(x))^{p_2/p_1} = (1-G_1(x))$. Hence the damage of Y_2 on Y_1 is confounding.

Definition: Two distribution functions $F(x)$ and $G(x)$ are said to be of the same type if there exist constants $a > 0$ and b such that

$$G(x) = F(ax+b) .$$

The three limiting types for the extreme (minimum) are the following distributions:

$$\Phi_{1,\alpha}(x) = \begin{cases} 1-\exp[1-(-x)^{-\alpha}] & x < 0 \\ 1 & x \geq 0 \end{cases} , \alpha > 0 ,$$

$$\Phi_{2,\alpha}(x) = \begin{cases} 0 & x \leq 0 \\ 1-\exp[-x^{\alpha}] & x > 0 \end{cases} , \alpha < 0 ,$$

and

$$\Lambda(x) = 1-\exp(-e^{x}) .$$

We now prove a lemma that will be required in the proof of the main theorem characterizing the three limiting types.

Lemma 1. Let $W(v)$ be a non-constant monotonic function with $0 \leq W(v) \leq 1$. Suppose that there exist constants $p_1 > 0$, $p_2 > 0$, A_1 and A_2 such that

(4) $$W^{p_i}(v+A_i) = W(v) \qquad i=1,2$$

and A_1/A_2 is irrational.

Then there exist constants a and b such that

$$W(v) = \exp(-e^{av+b}) ,$$

where a is positive or negative according as $W(v)$ is decreasing or increasing.

Proof: Let $E = \{v: v=mA_1+nA_2; m,n=\cdots -1,0,1,\ldots\}$. Then E is dense on the real line. Let $\psi(v) = \log(-\log W(v))$. Then $\psi(v)$ is monotonic. Let $D = \{v: \psi'(v)$ exists$\}$. Then by Lebesgue's theorem (e.g., Riesz and Sz.Nagy (1955), p. 5) ψ has a derivative almost everywhere and hence D is non-empty. From (4) we have

$$W^{p_1^m p_2^n}(v+mA_1+nA_2) = W(v) .$$

Hence

$$\psi(v+mA_1+nA_2)-\psi(v) = -\log p_1^{m_1} p_2^{n} .$$

Let for each $h = mA_1+nA_2 \epsilon E$, $p_h = p_1^m p_2^n$. Let $v_o \epsilon D$. Then

(5) $$\lim_{\substack{h\to o\\ h\epsilon E}} \frac{\psi(v_o+h)-\psi(v_o)}{h} = \lim_{\substack{h\to o\\ h\epsilon E}} - \frac{\log p_h}{h}$$

exists and is equal to a, say. Since the right hand side of (5) does not depend on v_o, $\psi'(v) = a$ for $v\epsilon D$. Also since relation (5) holds for general v, ψ is monotonic and E is dense, it follows that $\psi'(v)$ exists for all v and is equal to a. Hence $\psi(v) = av+b$, or

$W(v) = \exp(-e^{av+b})$. It is easy to see that a is positive or negative according as $W(v)$ is decreasing or increasing.

Theorem 2. Let the random variables X, Y_1 and Y_2 have different distribution functions $F(x)$, $F(a_1x+b_1)$ and $F(a_2x+b_2)$, respectively and be compatible with one another. Let $a_1 \geq 1$ and $a_2 \geq 1$. Let the damages of Y_1 on X and Y_2 of X be confounding. Then

(a) $a_1 > 1$ implies $a_2 > 1$, $\frac{b_1}{1-a_1} = \frac{b_2}{1-b_2}$.

$a_1 = 1$ implies $a_2 = 1$.

(b) If $a_1 > 1$ and $(\log a_1)/(\log a_2)$ is irrational

then $F(x)$ is of the type $\Phi_{1,\alpha}(x)$ or $\Phi_{2,\alpha}(x)$ according as $p_1 > 1$ or $p_1 < 1$, where p_1 is defined in (6) in the proof.

If $a_1 = 1$ and b_1/b_2 is irrational then $F(x)$ is of the type $\Lambda(x)$.

Proof: At the outset, we remark that, under the condition that the damages of Y_1 on X and of Y_2 on X are confounding, the assumptions $a_1 \geq 1$, $a_2 \geq 1$, are no loss of generality in view of remark 4.

Let $R(x) = 1-F(x)$. $R(x)$ is monotonic decreasing and $0 \leq R(x) \leq 1$. From theorem 1 there exist constants $p_1 > 0$, $p_2 > 0$ such that

$$R^{p_1}(a_1x+b_1) = R(x)$$
(6)
$$R^{p_2}(a_2x+b_2) = R(x) \text{ .}$$

Let $a_1 > 1$. Then $a_1x+b_1 \gtreqless x$ according as $x \gtreqless \frac{b_1}{1-a_1}$. Let $p_1 > 1$. Then

$$R(x) = R^{p_1}(a_1x+b_1) \leq R^{p_1}(x) \quad \text{for} \quad x \geq \frac{b_1}{1-a_1} .$$

Thus

(7) $$R(x) = 0 \quad \text{for} \quad x \geq \frac{b_1}{1-a_1} .$$

Now, suppose that $R(x_o) = 0$ for some $x_o < \frac{b_1}{1-a_1}$. Let $x < \frac{b_1}{1-a_1}$. Then there exists an integer k such that

$$a_1^k x_o + a_1^{k-1} b_1 + a_1^{k-2} b_1 + \cdots + b_1 = x_k < x \leq x_{k-1} = a_1^{k-1} x_o + a_1^{k-2} b_1 + a_1^{k-3} b_1 + \cdots + b_1 .$$

Thus $0 \leq R(x_o) = R^{p_1^k}(x_k) \geq R(x) \geq R^{p_1^{k-1}}(x_o) = R(x_o) = 0$.

Thus $R(x) = 0$ for $x < \frac{b_1}{1-a_1}$, which is impossible. Hence

(8) $$R(x) > 0 \quad \text{for} \quad x < \frac{b_1}{1-a_1} .$$

Similarly, if $p_1 < 1$ then

(9) $$R(x) = 1 \quad \text{for} \quad x \leq \frac{b_1}{1-a_1} \quad \text{and} \quad R(x) < 1 \quad \text{for} \quad x > \frac{b_1}{1-a_1} .$$

Let $p_1 > 1$. Now a_2 must be greater than 1. If not, then $a_2 = 1$ and $b_2 \neq 0$. Thus

$$R^{p_2}\left(\frac{b_1}{1-a_1} - b_2\right) = R\left(\frac{b_1}{1-a_1}\right) = R^{\frac{1}{p_2}}\left(\frac{b_1}{1-a_1} + b_2\right)$$

and

$$R(x) = 0 \quad \text{for} \quad x \quad \text{in} \quad \left[\frac{b_1}{1-a_1} - |b_2|, \frac{b_1}{1-a_1}\right] .$$

This is impossible because of (8). Hence $a_2 > 1$. Furthermore from (7), (8), and (9) we conclude that $p_2 > 1$ and $\frac{b_1}{1-a_1} = \frac{b_2}{1-a_2}$.

If $p_1 < 1$, then a similar argument shows that $a_2 > 1$, $p_2 < 1$ and $\frac{b_1}{1-a_1} = \frac{b_2}{1-a_2}$.

These also show that $a_1 = 1$ implies $a_2 = 1$. These now establish part (a) of the theorem.

Now, consider the case $a_1 > 1$, $p_1 > 1$. Let

$$S(x) = R(x + \frac{b_1}{1-a_1}) , \; T(x) = S(-x), \; W(v) = T(e^v) .$$

Then $W(v)$ is increasing and (6) can be rewritten as

$$(10) \qquad W^{p_i}(v+A_i) = W(v) \; , \; A_i = \log a_i \; , \quad i=1,2 .$$

Also A_1/A_2 is irrational. Hence, from Lemma 1, $W(v) = \exp(-e^{-\alpha v+\beta})$ where $\alpha > 0$. This shows that

$$F(x) = \begin{cases} 1-\exp\left\{-(-x + \frac{b_1}{1-a_1})^{-\alpha} \cdot e^{\beta}\right\} & x < \frac{b_1}{1-a_1} \\ 1 & x \geq \frac{b_1}{1-a_1} \end{cases} .$$

Hence $F(x)$ is of the type $\Phi_{1,\alpha}(x)$.

Let $a_1 > 1$, $p_1 < 1$. Let $S(x) = R(x + \frac{b_1}{1-a_1})$, $W(v) = S(e^v)$. Then $W(v)$ is decreasing and satisfies (10), where $A_i = \log a_i$, $i=1,2$. Also A_1/A_2 is irrational. From lemma 1, we have $W(v) = \exp(-e^{\alpha v+\beta})$ where $\alpha > 0$. Thus

$$F(x) = \begin{cases} 0 & x \leq \frac{b_1}{1-a_1} \\ 1-\exp\left\{-(x - \frac{b_1}{1-a_1})^{\alpha} \cdot e^{\beta}\right\} & x > \frac{b_1}{1-a_1} \end{cases} .$$

Hence $F(x)$ is of the type $\Phi_{2,\alpha}(x)$.

Let $a_1 = 1$. Let $R(v) = W(v)$. Then $W(v)$ is decreasing and satisfies (10) where $A_i = b_i$, $i=1,2$. Also b_1/b_2 is irrational. Hence from lemma 1, $W(v) = \exp(-e^{\alpha v+\beta})$ where $\alpha > 0$. Thus

$$F(x) = 1-\exp(-e^{\alpha v+\beta})$$

or $F(x)$ is of the type $\Lambda(x)$.

This completes the proof of part (b) of the theorem.

The condition $(\log a_1)(\log a_2)$ is irrational or b_1/b_2 is irrational is to ensure that the linear function a_1x+b_1 is essentially different from a_2x+b_2. The following remark describes the situation when this is not the case.

Remark 5. Let the distribution functions of X and Y be $F(x)$ and $F(x+b)$ and the random variables be compatible. Let the damage of Y on X be confounding. From theorem 1, a necessary and sufficient condition for the above is that there exists a constant $p > 0$ such that

$$(1-F(x+b))^p = F(x) \quad . \tag{11}$$

We now show that $F(x)$ need not be of the type $\Lambda(x)$. In fact any function $F(x)$ defined on $[0,b]$ to be a continuous increasing function with

$$(1-F(b))^p = (1-F(x)$$

can be extended to the whole real line to provide a solution to (11) and hence to the problem mentioned in the remark. One example different from $\Lambda(x)$ is got by letting $F(x)$ be a straight line on $[0,b]$, namely

$$F(x) = (1-\theta^{p^{-m}}) + (x-mb)[(1-\theta^{p^{\overline{-m+1}}}) - (1-\theta^{p^{-m}})], \; mb < x \leq (m+1)b,$$

$$m= \ldots -1,0,1,\ldots \; .$$

Remark 6. A remark similar to remark 5 holds if the distribution functions of X and Y were $F(x)$ and $F(ax)$ where $a \geq 1$. Also the general case, where these distribution functions are $F(x)$ and $F(ax+b)$, $a \geq 1$, can be reduced, as in the proof of part (a) of theorem 2, to the case of remark 5 or remark 6.

We now restate the above remarks and theorems in a different way in terms of a damage model involving several random variables.

Definition: The random variables $X_1, \ldots, X_k$ are said to be compatible if $P(X_i > X_j) > 0$ for $i \neq j$, $i,j=1,\ldots,k$.

Definition: The total damage among $X_1, X_2, \ldots, X_k$ $(k > 2)$ is confounding if the distribution of $Z = \min(X_1, X_2, \ldots, X_k)$, the conditional distribution of Z given $Z = X_1$, the conditional distribution of Z given $Z = X_2, \ldots$ and the conditional distribution of Z given $Z = X_k$ are all the same.

Theorem 3. Let the distribution functions of the compatible random variables $X_1, X_2, \ldots, X_k$ be $F_1(x), F_2(x), \ldots, F_k(x)$, respectively. Then the total damage among $X_1, X_2, \ldots, X_k$ is confounding if and only if there exist constants $p_2 > 0, \ldots, p_k > 0$ such that

$$(1-F_i(x))^{p_i} = 1-F_1(x) \qquad i=2,\ldots,k. \tag{12}$$

Theorem 4. Let the total damage among the compatible random variables $X_1, X_2, \ldots, X_k$ be confounding. Let the distribution functions of $X_1, X_2, \ldots, X_k$ be of the same type, namely $F(x), F(a_2x+b_2), \ldots, F(a_kx+b_k)$, where, without loss of generality $a_2 \geq 1, \ldots, a_k \geq 1$.

If $a_2 > 1$ then $a_i > 1$, $\frac{b_2}{1-a_2} = \frac{b_i}{1-a_i}$, $i=3,\ldots,k$. If, further, $(\log a_i)/(\log a_j)$ is irrational for some $i, j \geq 2$ then $F(x)$ is of type $\Phi_{1,\alpha}(x)$ or $\Phi_{2,\alpha}(x)$ according as p_2 in (12) is > 1 or < 1.

If $a_1 = 1$ then $a_i = 1$, $i=3,\ldots,k$. If, further b_i/b_j is irrational for some $i, j \geq 2$ then $F(x)$ is of the type $\Lambda(x)$.

Finally, we wish to remark that we would obtain analogous results if we consider the damage model obtained by putting max in place of min, in the above discussion.

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