APPLIED MATHEMATICS AND STATISTICS LABORATORY
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THE SUPREMUM AND INFIMUM OF THE POISSON PROCESS

By
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1. Introduction: Let $\{X(t);\ t \geq 0\}$ be a separable Poisson process with shift such that

$$\log E(e^{i\omega X(t)}) = -it\omega\alpha + \lambda t(e^{i\omega} - 1) \tag{1}$$

for all real ω, and $\alpha, \lambda > 0$. Set

$$\sigma(x,T) = P[\sup_{0 \leq t \leq T} X(t) \leq x]$$

The task of obtaining $\sigma(x,T)$ explicitly for general stochastic processes is intrinsically difficult. However, Baxter and Donsker [1], following the methods and results of Spitzer, have obtained the double Laplace transform of $\sigma(x,T)$ for processes with stationary and independent increments. Their result as it pertains to the Poisson process is as follows.

Theorem: Let $\{X(t);\ t \geq 0\}$ be a separable process satisfying $X(0) = 0$ a.s. and

$$\log E(e^{i\omega X(t)}) = t\psi(\omega)$$

for all $t \geq 0$ where $\exp(\psi(\omega))$ is the Levy-Khintchine representation of the characteristic function of an infinitely divisible distribution. If $\psi(\omega)$ is complex and for some $\delta > 0$

$$\int_{-\delta}^{\delta} \left| \frac{\psi(\omega)}{\omega} \right| d\omega < \infty$$

then for all $u, v \geq 0$

$$(2) \qquad u\int_0^\infty \int_{0-}^\infty e^{-uT-vx}\, d_x\sigma(x,T)dT = \exp\left\{\frac{1}{2\pi}\int_u^\infty \int_{-\infty}^\infty \frac{v}{\omega(\omega-iv)}\ \frac{\psi(\omega)}{s[s-\psi(\omega)]}\right\} d\omega ds$$

Theoretically, therefore, to obtain $\sigma(x,T)$ explicitly, one should evaluate the double integral on the right hand side of (2) and then perform a double inversion on it. For most cases this is virtually impossible except by numerical methods. Baxter and Donsker, however, have evaluated the right hand side of (2) for several important cases. Moreover, for the Gaussian process and for the process determined by coin tossing at random times, they were able to make the inversions.

For the Poisson process, Baxter and Donsker showed that

$$(3) \qquad u\int_0^\infty \int_{0-}^\infty e^{-uT-vx}\, d_x\sigma(x,T)dT = (1-v/s_u)\ [1-\psi(iv)/u]^{-1}$$

where s_u satisfies $\mathcal{I}m(s_u) \geq 0$ and

$$u = \alpha\, s_u + \lambda(e^{-s_u} - 1)$$

From (3) they obtained $\sigma(x,+\infty)$. It is the purpose of this paper to obtain $\sigma(x,T)$ for finite T. In section 4, the corresponding equation to (3), as well as the exact distribution for the infimum of this process, are derived. Applications of these results to queuing theory are given in section 5. First of all, a lemma with applications to the theory of distribution-free statistics, is proven.

2. A Lemma: Let $X_1, X_2, \ldots, X_n$ be independent random variables on a common measure space $(\Omega, \mathcal{O}\!\mathcal{l}, P)$ such that $P[X_j \leq x] = x$ for all $0 \leq x \leq 1$. Define U_j as the j^{th} smallest component of $(X_1, X_2, \ldots, X_n)$.

Therefore U_j is well defined a.s. Define for all real x, a and integral n

$$F(x:a,n) = P[\max_{1 \leq i \leq n} (ai-U_i) \leq x]$$

<u>Lemma 1</u>. For $0 \leq a \leq 1$, $0 \leq na-x < 1$,

$$(4) \qquad F(x:a,n) = (1+x-na) \sum_{j=0}^{[x/a]} \binom{n}{j} (ja-x)^j (1+x-ja)^{n-j-1}$$

where $[y]$, the greatest integer contained in y , is a left continuous function. When $na-x \geq 1$ or $na-x < 0$, $F(x:a,n)$ is equal to 0 or 1 respectively.

Proof: It may be shown that whenever $0 \leq na-x < 1$,

$$F(x:a,n) = n! \int_{na-x}^{1} dz_n \int_{(n-1)a-x}^{z_n} dz_{n-1} \cdots \int_{ka-x}^{z_{k+1}} dz_k \int_0^{z_k} dz_{k-1} \cdots \int_0^{z_2} dz_1$$

where $k = [x/a] + 1$. Clearly

$$\int_0^{z_k} dz_{k-1} \cdots \int_0^{z_2} dz_1 = \frac{z_k^{k-1}}{(k-1)!}$$

Define

$$I_j = n! \int_{na-x}^{1} dz_n \cdots \int_{ja-x}^{z_{j+1}} \frac{z_j^{j-1}}{(j-1)!} dz_j$$

and

$$A_j = n! \int_{na-x}^{1} dz_n \cdots \int_{ja-x}^{z_{j+1}} dz_j$$

for $j=k, k+1, \ldots, n$. Therefore $F(x:a,n) = I_k$. It is easily checked that

(5)
$$I_j = I_{j+1} - \frac{(ja-x)^j}{j!} A_{j+1}$$

Under the change of variables, $v_i = z_i - ja + x$, $i=j+1, \ldots, n$, one obtains

$$A_{j+1} = n! \int_{(n-j)a}^{1-ja+x} dv_n \cdots \int_{2a}^{v_{j+3}} dv_{j+2} \int_a^{v_{j+2}} dv_{j+1}$$

$$= n! \int_{(n-j)a}^{1-ja+x} \left[\frac{v_n^{\,n-j-1}}{(n-j-1)!} - a\frac{v_n^{\,n-j-2}}{(n-j-2)!} \right] dv_n$$

$$= \frac{n!}{(n-j)!} (1-ja+x)^{n-j-1} (1-na+x)$$

Therefore (5) becomes

$$I_j - I_{j+1} = -(1-na+x) \binom{n}{j} (ja-x)^j (1-ja+x)^{n-j-1}$$

Consequently, by addition

$$F(x:a,n) = I_k = I_n - (1-na+x) \sum_{j=k}^{n-1} \binom{n}{j} (ja-x)^j (1-ja+x)^{n-j-1}$$

$$= 1 - (1-na+x) \sum_{j=k}^{n} \binom{n}{j} (ja-x)^j (1-ja+x)^{n-j-1}$$

$$= (1-na+x) \sum_{j=0}^{k-1} \binom{n}{j} (ja-x)^j (1-ja+x)^{n-j-1}$$

as desired. The last step follows from Lemma 1 of [2]. The remainder of Lemma 1 may be shown directly from the definition.

3. The Derivation of $\sigma(x,T)$. Let $\{Y(t),\ t\geq 0\}$ be a Poisson process with parameter $\lambda > 0$; that is

$$E\{e^{i\omega\, Y(t)}\} = e^{\lambda t(e^{i\omega}-1)}$$

Write $X(t) = Y(t)-\alpha t$. Then

$$\sigma(x,T) = P[\sup_{0\leq t\leq T} X(t) \leq x] = P[Y(t) \leq \alpha t + x;\ 0 \leq t \leq T]$$

It is well known (cf. [3] chap. VIII) that the conditional distribution of the first n discontinuity points of the Poisson process given that there were n such points in $(0,T)$, is the distribution of $(TU_1, TU_2, \ldots, TU_n)$ where the U_i's are as defined above. Using this fact, one may write

$$\sigma(x,T) = \sum_{n=0}^{[\alpha T+x]} P[\max_{1\leq i\leq n} (i/\alpha T-U_i) \leq x/\alpha T]\ e^{-\lambda T} \frac{(\lambda T)^n}{n!}$$

Evaluating the summands by means of Lemma 1 gives

$$\sigma(x,T) = \sum_{n=0}^{[x]} \frac{e^{-\lambda T}(\lambda T)^n}{n!} + \sum_{n=[x]+1}^{[\alpha T+x]} e^{-\lambda T} \frac{(\lambda T)^n}{n!} (\alpha T+x-n)\ (\alpha T)^{-n}$$

$$\cdot \sum_{j=0}^{[x]} \binom{n}{j} (j-x)^j\ (\alpha T+x-j)^{n-j-1}$$

We have therefore proven the following

Theorem 1. Let $\{Y(t)\ ,\ t\geq 0\}$ be a Poisson process with parameter $\lambda>0$. Then for all $\alpha,\ x>0$

$$(6) \qquad \sigma(x,T) = \sum_{n=0}^{[\alpha T+x]} \frac{(\alpha T+x-n)(\lambda/\alpha)^n}{n!} \sum_{j=0}^{[x]} \binom{n}{j} (j-x)^j\ (\alpha T+x-j)^{n-j-1}$$

where $\binom{n}{j} = 0$ for $j > n$.

Corollary: For all $T > 0$, $\alpha > 0$

$$\sigma(0,T) = e^{-\lambda T} \sum_{n=0}^{[\alpha T]} \frac{(\lambda T)^n}{n!} (1-n/\alpha T)$$

The limiting case of $\sigma(x,+\infty)$ may be obtained from Theorem 1 by an application of the Central Limit Theorem. More specifically, by the Central Limit Theorem, for $j \geq [x]$

$$\lim_{T \to \infty} \sum_{n=j}^{[\alpha T+x]} e^{-\lambda T} \frac{\{\lambda T+(x-j)\lambda/\alpha\}^{n-j-1}}{(n-j)!} \{\lambda T+(x-n)\lambda/\alpha\}$$

$$= (1-\lambda/\alpha)\, e^{\lambda(x-j)/\alpha} \lim_{T \to \infty} \sum_{k=0}^{[\alpha T+x]} e^{-\lambda T-(x-j)\lambda/\alpha} \frac{\{\lambda T+(x-j)/\alpha\}^k}{k!}$$

$$= \begin{cases} (1-\lambda/\alpha)\, e^{\lambda(x-j)/\alpha} & \text{if } \lambda < \alpha \\ 0 & \text{if } \lambda \geq \alpha \end{cases}$$

Therefore, from Theorem 1, for $x \geq 0$

(7) $$\sigma(x,+\infty) = (1-\lambda/\alpha) \sum_{j=0}^{[x]} \left(\frac{\lambda}{\alpha}\right)^j \frac{(j-x)^j}{j!} e^{-\lambda(j-x)/\alpha}$$

when $\lambda < \alpha$ and is equal to zero otherwise. This formula disagrees with (4.15) of [1]. For $x=0$, (7) becomes

$$\sigma(0,+\infty) = \begin{cases} 1 - \lambda/\alpha & \text{if } \lambda/\alpha > 0 \\ 0 & \text{otherwise} \end{cases}$$

The expression (7) has also been obtained by Breakwell [7] who has computed a constant multiple of (7) for several values of the parameters.

4. The Infimum of $\{X(t),\ t \geq 0\}$. It is also of interest to study the infimum of the deviations of the Poisson process about the line αt .

Since

$$\inf_{0\leq t\leq T} X(t) = -\sup_{0\leq t\leq T} \{-X(t)\}$$

a double generating function, $\Phi(u,v)$ say, may be obtained for the infimum by the same methods as used for the supremum by Baxter and Donsker [1]. The logarithm of the characteristic function of $-X(t)$ is clearly $t\psi(-\omega)$ where for all complex z

$$\psi(z) = -i\alpha z + \lambda(e^{iz}-1)$$

as in (1). It follows from (2) that

$$u\Phi(u,v) = \exp\left\{\frac{1}{2\pi}\int_u^{\infty}\int_{-\infty}^{\infty} \frac{v}{\omega(\omega+iv)}\,\frac{\psi(\omega)}{s[s-\psi(\omega)]}\,d\omega ds\right\}$$

By an application of Rouché's theorem, it may be shown that for all $s > 0$, $\psi(z)-s$ has as many roots in the upper half plane, $|z| \geq 0$, as $h(z)=i\alpha z+\lambda+s$, namely one. Denote this root by iy_s; that is

$$\psi(iy_s) = s = \alpha y_s + \lambda(e^{-y_s} - 1)$$

Since $\overline{iy_s}$ is also a root, its uniqueness implies that y_s is real. Moreover the root is simple. Straightforward integration in the upper half plane yields

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{v}{\omega(\omega+iv)}\,\frac{\psi(\omega)}{s[s-\psi(\omega)]}\,d\omega ds = -\frac{v}{y_s(y_s+v)}\,\frac{dy_s}{ds}$$

Consequently

$$u\Phi(u,v) = (1 + v/y_u)^{-1}$$

Although unable to make a double inversion of Φ , we considered

$$\lim_{u \to o} u\,\Phi(u,v) = \int_o^\infty e^{-vx}\, d_x\, P[-\inf_{0 \geq t < \infty} X(t) \leq x]$$

$$= (1 + v/y_o)^{-1}$$

Under the assumption $\alpha < \lambda$, y_o is the unique positive root of the real function $\alpha y + \lambda(e^{-y} - 1)$. When $\alpha \geq \lambda$, $y_o = 0$ and the above limit is defined to be zero. It follows, in particular, that for $\alpha < \lambda$, the infimum over $[o,\infty)$ of $X(t)$ has an exponential distribution with parameter y_o .

It is clear that the above method may be used in general to obtain the distribution of the infimum of any process with stationary and independent increments from the theory of Baxter and Donsker concerning the supremum.

In the following, the explicit distribution function of the infimum over finite intervals of the Poisson process is derived. Moreover, an expression for the distribution function of the infimum over $[o,\infty)$ is obtained. It is shown by a second method that this latter distribution is exponential. The advantage of the second method is that the parameter y_o as defined above is obtained explicitly.

For $x \leq 0$ set

$$\mu(x,T) = P[\inf_{0 \leq t \leq T} X(t) \leq x]$$

$$= 1 - P[Y(t) > \alpha t + x \ , \ 0 \leq t \leq T]$$

for all $T \geq 0$. It is clear that whenever $\alpha T + x \leq 0$ $\mu(x,T) = 0$. Suppose $\alpha T + x > 0$. Then by a similar argument to that used in section 3,

$$1 - \mu(x,T) = \sum_{n=0}^{\infty} e^{-\lambda T} \frac{(\lambda T)^n}{n!} P[TU_i < \frac{i-1-x}{\alpha} \ , \ i=1, 2, \ldots, n]$$

$$= \sum_{n=[\alpha T+x]+1}^{\infty} e^{-\lambda T} \frac{(\lambda T)^n}{n!} P[\max_{1\leq i\leq n} (i/\alpha T - U_i) \leq \frac{n-x}{\alpha T} - 1]$$

since the distribution functions of $(U_1, U_2, \ldots, U_n)$ and $(1-U_n, 1-U_{n-1}, \ldots, 1-U_1)$ are the same. Therefore, by Lemma 1, one obtains

$$\mu(x,T) = 1 + x \sum_{n=K+1}^{\infty} e^{-\lambda T} \frac{(\lambda/\alpha)^n}{n!} \sum_{j=0}^{n-1-K} \binom{n}{j} (j-n+x+\alpha T)^j (n-j-x)^{n-j-1}$$

where $K = [\alpha T+x]$. Hence, upon rearranging the summations, we have proven

Theorem 2: For all $x \leq 0$, α, $T \geq 0$, the distribution function of $\inf_{0\leq t\leq T} X(t)$, $\mu(x,T)$, is given by

$$\mu(x,T) = \sum_{n=0}^{K} e^{-\lambda T} \frac{(\lambda T)^n}{n!} - x e^{-\lambda T} \sum_{r=0}^{K} \frac{(r-x)^{r-1}}{r!} \sum_{n=K+1}^{\infty} \frac{(\lambda/\alpha)^n}{(n-r)!} (x+\alpha T-r)^{n-r}$$

To obtain $\mu(x,+\infty)$, one needs the fact that for all $r \geq 0$

(8) $$\lim_{T\to\infty} \sum_{s=K+1-r}^{\infty} e^{-\lambda(x-r)/\alpha-\lambda T} \frac{\{\lambda(x-r)/\alpha+\lambda T\}^s}{s!} = \begin{cases} 1 & \text{if } \alpha < \lambda \\ 1/2 & \text{if } \alpha = \lambda \\ 0 & \text{if } \alpha > \lambda \end{cases}$$

a result which follows from the Central Limit Theorem as applied to the Poisson distribution. Therefore, whenever $\alpha > \lambda$, $\mu(x,+\infty) = 1$, and whenever $\alpha < \lambda$

(9) $$\mu(x,+\infty) = -x \sum_{r=0}^{\infty} e^{\lambda(x-r)/\alpha} (r-x)^{r-1} \frac{(\lambda/\alpha)^r}{r!}$$

since all the summands are positive. This distribution shall be shown to be exponential over $(-\infty, o]$. Specifically, we shall prove

Theorem 3: For all $x \leq 0$, $\lambda > \alpha > 0$,

(10) $$\log \mu(x,+\infty) = x\lambda\alpha^{-1} \{1-\mu(-1,+\infty)\}$$

It will be convenient to define for all real x and $\beta \leq e^{-1}$

$$(11) \qquad f(x,\beta) = x \sum_{j=0}^{\infty} \frac{\beta^j}{j!} (j+x)^{j-1}$$

Since $f(x,\beta)$ is a power series in β we have

$$(12) \qquad f(x,\beta)\, f(y,\beta) = xy \sum_{n=0}^{\infty} \frac{\beta^n}{n!} \sum_{j=0}^{n} \frac{(j+x)^{j-1}}{j!} \frac{(n-j+y)^{n-j-1}}{(n-j)!}$$

The inner summation may be written as

$$(n+x)^{-1} \left\{ \sum_{j=0}^{n} \frac{(j+x)^j}{j!} \frac{(n-j+y)^{n-j-1}}{(n-j)!} + \sum_{j=0}^{n-1} \frac{(j+x)^{j-1}}{j!} \frac{(n-j+y)^{n-j-1}}{(n-j-1)!} \right\}$$

$$= \frac{(n+x)^{-1}}{n!} \left\{ \frac{(n+y+x)^n}{y} - \frac{n(n+y+x)^{n-1}}{x} \right\}$$

the latter following as a consequence of Lemma 1 of [2]. Upon simplifying and substituting into (12), one obtains

$$f(x,\beta)\, f(y,\beta) = f(x+y,\beta)$$

Since $f(x,\beta)$ is a continuous function of x when $f(0,\beta)$ is defined to equal 1, it is known that

$$(13) \qquad f(x,\beta) = e^{xg(\beta)}$$

for some function $g(\beta)$ independent of x. For a fixed $\beta < e^{-1}$, $g(\beta)$ may be obtained by differentiation w·r·t x and taking limits as $x \to 0$. That is

$$\frac{\partial}{\partial x} f(x,\beta) = x \sum_{j=0}^{\infty} \frac{\beta^j}{j!} (j+x)^{j-1} \left\{ \frac{1}{x} + \frac{j-1}{j+x} \right\}$$

and so

$$g(\beta) = \lim_{x \to 0} \frac{\partial}{\partial x} f(x,\beta) = \beta f(1,\beta) \tag{14}$$

Setting $v = \lambda/\alpha$ and $\beta = ve^{-v}$ in (9) and (11) gives

$$\mu(x,+\infty) = e^{vx} f(-x, ve^{-v}) = \exp\{xv - xve^{-v} f(1, ve^{-v})\}$$

$$= \exp\{vx - vx\, \mu(-1,+\infty)\}$$

which is the desired result.

Upon setting $x = -1$ in (10), one obtains

$$\log \mu(-1,+\infty) = -v\{1-\mu(-1,+\infty)\}$$

or equivalently, one has shown that $v\{1-\mu(-1,+\infty)\}$ is a non-negative root of the equation

$$\lambda(e^{-z}-1) + \alpha z = 0 \tag{15}$$

That is, in the notation of (3), $v\{1-\mu(-1,+\infty)\} = s_o$.

The function $\mu(x,+\infty)$ is a special case of the ruin function studied in the theory of collective risk which has already been shown to have an exponential form e^{Rx} where R is the unique solution of (15), (cf. Cramér [5]). The result for $\alpha < \lambda$ contained in Theorem 3 is new in that it gives an explicit expression for R .

It remains to evaluate $\mu(x,+\infty)$ when $\alpha = \lambda$. From Theorem 2 and (8) it follows that for $\alpha = \lambda$

$$\mu(x,+\infty) = \frac{1}{2} + \frac{1}{2} \exp\{x - xe^{-1} f(1, e^{-1})\}$$

However, in this case $1-e^{-1} f(1,e^{-1})$ must be a solution of (15) with $\lambda = \alpha$. It then follows that $1-e^{-1} f(1,e^{-1}) = 0$, that is, $f(1,e^{-1}) = e$. In summary, therefore,

$$\mu(x,+\infty) = \begin{cases} 1 & \text{if } \alpha \geq \lambda \\ \exp\left\{x\lambda\alpha^{-1} [1-\mu(-1,+\infty)]\right\} & \text{if } \alpha < \lambda \end{cases}$$

The identity contained in (13) and (14) is of interest in its own right, and so it is given separately in

Lemma 2: For all real x and $\beta \leq e^{-1}$

$$\log\left\{x \sum_{j=0}^{\infty} \frac{\beta^j}{j!} (j+x)^{j-1}\right\} = x\beta \sum_{j=0}^{\infty} \frac{\beta^j}{j!} (j+1)^{j-1}$$

the restriction $\beta \leq e^{-1}$ sufficing to make both series convergent.

5. Applications to Queuing Theory: Suppose that customers are arriving at times n/α, $n=0,1,2,\ldots$ and that the service time for the j^{th} customer, S_j say, is exponentially distributed with expectation $1/\lambda$. It is of importance in queuing theory to determine the distribution of the busy period of the server under the initial condition that there were k people in the queue. To this end one must compute

$B(T|k) \equiv P[\text{server is busy throughout } (0,T] \mid k \text{ people in the queue at } t=0]$

Since additional customers are arriving at times n/α, $n = 1,2,\ldots$ we have

$$B(T|k) = P[S_1 + S_2 + \ldots + S_{i+k} \geq (i+1)/\alpha,\ 0 \leq i \leq \alpha T]$$

$$= P[Y(t) \leq \alpha t+k-1,\ 0 \leq t \leq T]$$

where $Y(t)$ is a Poisson process with parameter λ. Therefore, $B(T|k) = \sigma(k-1,T)$. Define T_k as the time until the server is free under the condition that there are $k \geq 1$ in line at time $t = 0$. Let G_k be the distribution function of T_k. Then clearly $T_k > 0$ a.s. and for all $t \geq 0$

$$G_k(t) = 1 - B(t|k) = 1 - \sigma(k-1,t)$$

which may then be evaluated by Theorem 1. In particular T_1 represents the total busy period of the server and its distribution function is $G_1(t) = 1-\sigma(0,t)$ which is given by the Corollary to Theorem 1.

A second application is of Theorem 2 to the queuing model in which the service times are constant and equal to $1/\alpha$ and the times between arrivals are independent random variables distributed exponentially with expectation $1/\lambda$. Let t_i denote the arrival time of the i^{th} person after $t=0$. As in the above let T_k denote the time until the server is free measured from $t=0$ when it is assumed that at $t=0$, there are k people in the queue and the server is just beginning service. Thus if G_k denotes the distribution function of T_k, $G_k(t) = 0$ for $t < 0$, and for $t \geq 0$

$$G_k(t) = 1 - P[t_i \leq \frac{i+k-1}{\alpha}, \ i = 1,2, \ldots, N_t]$$

where N_t is the number of customers arriving in $(0,t]$. Therefore,

$$G_k(t) = 1 - P[Y(u) \geq \alpha u-k, \ 0 \leq u \leq t] = \mu(-k,t)$$

which may be evaluated by Theorem 2. In particular T_1 represents the total busy period of the server and its distribution function is given by $\mu(-1,t)$.

Of special interest is $1-G_1(+\infty)$, which is the probability of the server being busy for an infinite length of time, or equivalently, of there always being a waiting line. Since $1 - G_1(+\infty) = 1 - \mu(-1,+\infty)$ we have by section 4, that $1 - G_1(+\infty) = 0$ whenever $\alpha \geq \lambda$, and

$$1 - G_1(+\infty) = 1 - e^{-\lambda/\alpha} \sum_{j=0}^{\infty} e^{-j\lambda/\alpha} (j+1)^{j-1} \frac{(\lambda/\alpha)^j}{j!}$$

whenever $\alpha < \lambda$. That $G_1(\cdot)$ is a proper distribution function only when $\alpha \geq \lambda$ is in keeping with the known result that the recurrent event, "the server is not busy" is ergodic whenever $\alpha > \lambda$, null recurrent whenever $\alpha = \lambda$ and transient whenever $\alpha < \lambda$. (cf. Lindley [5])

6. Applications of Lemma 1. The result given by Lemma 1 is of use in the theory of distribution-free statistics. The special case of (4) with $a = 1/n$ is the distribution function of the D_n^+- statistic. This special case is known, having been obtained by several authors (see e.g. [7]).

A slightly modified version of the D_n^+- statistic is

$$\max_{1\leq i\leq n} \left(\frac{i}{n+1} - U_i\right) = C_n^+ \text{ say}$$

This latter statistic has the same asymtotic properties as D_n^+ as well as having other small sample properties. For example $E(\frac{i}{n+1} - U_i) = 0$ for all i. Moreover, setting $U_{n+1} = 1$, $U_o = 0$ one may write

$$W_i = \frac{1}{n+1} - U_i + U_{i-1} \qquad i = 1,2, \ldots, n+1$$

and

$$S_j = \sum_{i=1}^{j} W_i \qquad j = 1,2, \ldots, n+1$$

It is known that the distribution of $(W_1, \ldots, W_n)$ is the same as that of any permutation of it. Therefore by a result of Andersen [8]

$$P[C_n^+ = \frac{j}{n+1} - U_j] = \frac{1}{n+1}, \; j = 0, 1, \ldots, n$$

That is to say, the probability that the maximum should occur at the j^{th} observation is independent of j. This is not so for D_n^+ as was shown in [2]. The distribution function of C_n^+, i.e., $P[C_n^+ \leq x]$, is given by $F(x: \frac{1}{n+1}, n)$ for $x \geq -\frac{1}{n+1}$ and is equal to zero for $x < -\frac{1}{n+1}$.

Lemma 1 may also be used to obtain the power of the D_n^+ or C_n^+ tests against alternatives of the form $G_c(x) = cx$ for all $x \epsilon [0,1/c]$. That is to say one may obtain, for example, the power of D_n^+ against G_c, namely

$$P[\max_{1 \leq i \leq n} (i/n - Z_1) \leq x] \equiv P[\max_{1 \leq i \leq n} (i/cn - U_i) \leq x/c] \tag{16}$$

The latter probability may be evaluated by Lemma 1 and is equal to the first probability where Z_i is the i^{th} smallest component of $(V_1, \ldots, V_n)$ in which the V_i's are mutually independent random variables with the common distribution function G_c. Similarly the power of D_n^+ or C_n^+ against alternatives of the form $G_{b,c}(x) = b + cx$ for all $x \epsilon [0,1/c - b/c]$ and $= 0$ for $x < 0$, may be expressed as a sum of a finite number of terms of the form (16). This generalization of (16) has recently been studied by Chapman [9] for the case $b+c = 1$.

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