APPLIED MATHEMATICS AND STATISTICS LABORATORIES
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SEQUENTIAL TESTING OF TRUNCATION PARAMETERS (LARGE SAMPLE THEORY)

By

GIDEON SCHWARZ

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1. Introduction.

In many statistical problems the range (i.e., the point set of positive density) of the random variable is prescribed. The simplest examples of such problems arise when the possible probability distributions form a Koopman-Darmois family [1]. Many of the more familiar distributions fall in this family. There are, however, various problems in which the range is not specified. In some of these, the determination of the range is the main object of the experiment.

For example, when an experiment is designed in order to determine the incubation time of a disease which cannot be induced in the laboratory, the experimenter may have to resort to exposing the subjects in a disease-carrying environment, and measure the time that elapses till they develop symptoms. Assuming an exponential distribution of the time from the start of exposure till infection takes place, and a constant incubation time, the measurements will be sums of an exponential random variable and an unknown constant R. They will be distributed with positive density on the interval $[R,\infty)$. A parameter that governs, like R in this example, the set of positive density, we call a "truncation parameter."

In a previous paper [2] the asymptotic shape approach to large sample theory of sequential testing was developed for families of distributions of the Koopman-Darmois type. In this paper, similar results are derived for the case of hypotheses about a truncation parameter. The results cover the case of a fixed distribution truncated at an unknown point, as well as the case of a distribution that belongs to a Koopman-Darmois family and is truncated at an unknown point. In the latter case, the Koopman-Darmois parameters appear as nuisance parameters.

In order to bring the paper closer to being self-contained, a survey of [2] is included in the next section. However, for a fuller motivated exposition, and for most of the proof of theorem II, we refer the reader to [2]. A special case of theorem II of [2] was proved in [3]. A definition of "asymptotic shape" can also be found there.

2. General Concepts.

The concept of "asymptotic shape" for some parametrized families of regions in n-dimensional Euclidean vector space is introduced in [2]. If a family of regions $Q(r)$, defined for positive r, has the property that for some real function $q(r)$ the family of homothetically transformed regions $\frac{1}{q(r)} Q(r)$ tends to a limiting region Q_o as r tends to zero, we call Q_o the asymptotic shape of $Q(r)$, and write

$$Q(r) = q(r)\, Q_o + o(q(r)) \quad (0)$$

The main result of [2] consists of an explicit formula for B_o, the asymptotic shape of the Bayes sampling region $B(c,W)$ for fixed a priori distribution W, and cost of sampling c approaching zero. The region $B(c,W)$ is defined in the $p+1$ dimensional space with coordinates $(n, S_{n1}, S_{n2}, \ldots, S_{np})$ where $S_{nk} = \sum_{i=1}^{n} z_k(X_i)$, $k = 1, 2, \ldots, p$ are the sufficient statistics for n observations.

In the case of truncation, the vector of sufficient statistics has the additional component $m_n = \max_{1 \leq i \leq n} X_i$; however, since the method of asymptotic shapes requires the statistics to have expectations that grow asymptotically linearly with n, we shall use $n\, m_n$ instead of m_n, and the coordinates in our $p+2$-dimensional space are therefore $(n, S_{n1}, \ldots, S_{np}, n\, m_n)$.

As in [2], the asymptotic shape B_o of the Bayes regions can be expressed in terms of two likelihood ratio statistics.

3. Testing a Truncation Parameter.

The distribution of the random variable X is assumed to belong to a $p+1$ parametric family of distribution with parameters $(\theta_1, \theta_2, \ldots, \theta_p, \tau)$, or for short $(\underset{\sim}{\theta}, \tau)$. τ is the truncation parameter, and $\underset{\sim}{\theta}$ enters the distribution through a Koopman-Darmois type density; that is, there exists a measure μ on the real line with respect to which the distribution of X has a density given by

$$(1)\qquad \begin{cases} e^{\Sigma \theta_i z_i(x) - b(\underset{\sim}{\theta},\tau)} & \text{when} \quad x \leq \tau \\ 0 & \text{when} \quad x > \tau . \end{cases}$$

Here $\underset{\sim}{z} = (z_1(x), \ldots , z_p(x))$ is a vector of dimension p, and

$$(2)\qquad b(\underset{\sim}{\theta},\tau) = \log \int_{-\infty}^{\tau} e^{\Sigma \theta_i z_i(x)} d\mu(x) .$$

There are two hypotheses given, H_o and H_1, and we assume there are two numbers $\tau_o < \tau_1$, such that $\tau \leq \tau_o$ for all $(\underset{\sim}{\theta},\tau)$ in H_o, and $\tau \geq \tau_1$, for all $(\underset{\sim}{\theta},\tau)$ in H_1.

From (2), $b(\underset{\sim}{\theta},\tau)$ is clearly increasing in τ, and $b(\underset{\sim}{\theta},\tau) - b(\underset{\sim}{\theta},\tau_o)$ is therefore nonnegative throughout H_1. We now impose the restriction that it will be bounded away from zero there, that is, for some $\Delta > 0$,

$$(3)\qquad b(\underset{\sim}{\theta},\tau) - b(\underset{\sim}{\theta},\tau_o) > \Delta \quad \text{for all} \quad (\underset{\sim}{\theta},\tau) \in H_1 .$$

We prove a lemma about a class of fixed-sample-size procedures $\mathcal{J}(r)$.

<u>Lemma A</u>: Let r be positive, and denote by $\mathcal{J}(r)$ the procedure that consists of taking N observations, with N the first positive integer greater than $\Delta^{-1} \log r^{-1}$ and deciding H_o or H_1 according to whether $\max_{1\leq k \leq N} X_k \leq \tau_o$ or $\max_{1\leq k\leq N} X_k > \tau_o$ respectively. Then, the probability that the procedure leads to a wrong decision is less than r throughout $H_o \cup H_1$.

<u>Proof</u>: For $(\underset{\sim}{\theta},\tau) \in H_o$, $\tau \leq \tau_o$ and the probability of a wrong decision is zero. For $(\underset{\sim}{\theta},\tau) \in H_1$, we have $\tau \geq \tau_1 > \tau_o$, and hence

$$\text{(4)} \quad \begin{aligned} \operatorname{Prob}\{\max X_k \leq \tau_o\} &= \left(\int_{x \leq \tau_o} e^{\Sigma \theta_i z_i(x) - b(\underset{\sim}{\theta},\tau)} d\mu(x) \right)^N \\ &= e^{-N(b(\underset{\sim}{\theta},\tau) - b(\underset{\sim}{\theta},\tau_o))} < e^{-N\Delta} < e^{-(\Delta^{-1} \log r^{-1})\Delta} = r \,. \end{aligned}$$

In order to give meaning to a Bayes formulation of the testing problem, we now introduce the loss and cost structure by assuming that the loss for making the wrong decision is given by a bounded function $L(\underset{\sim}{\theta},\tau)$ defined and positive throughout $H_o \cup H_1$, and the cost of observing is c per observation. We assume the utility unit chosen such that $L(\underset{\sim}{\theta},\tau) \leq 1$. We also assume a given <u>a priori</u> distribution W that dominates the Lebesgue measure on the parameter space Ω . The set Ω is a subset of Euclidean $p+1$ space, and we have $H_o \cup H_1 \subset \Omega$. The (possibly empty) set $\Omega - (H_o \cup H_1)$ is called the "indifference region", and it consists of all parameter points which are possible under W , but at which none of the two decisions leads to any loss.

We can now formulate and prove a theorem connecting the Bayes sampling procedure with the a posteriori risk.

<u>Theorem I: If at any stage of sampling the a posteriori risks of stopping and deciding</u> H_o <u>or</u> H_1 <u>are both at least</u> $(2 + \Delta^{-1} \log c^{-1})\, c$ <u>than any Bayes procedure will lead to taking another observation.</u>

Proof: The theorem will be proved if we show that there is one way of going on sampling for which the combined risk and cost are less than $(2+\Delta^{-1}\log c^{-1})c$. We shall show that this is the case for the procedure $\mathcal{J}(c)$ defined in the statement of lemma A. First, by the definition of $\mathcal{J}(c)$, the probability that it will lead to saying H_1 when H_o is true is equal to zero. Second, by the lemma, the probability of an error under H_1 is less than c. As $L(\underset{\sim}{\theta},\tau) \leq 1$, the expected loss for error is therefore less than c, and as the sample size of $\mathcal{J}(c)$ is at most $(\Delta^{-1}\log c^{-1}) + 1$, the expected loss and the cost add up to a number less than $c + [\Delta^{-1}\log c^{-1}) + 1]c =$
$= (2 + \Delta^{-1}\log c^{-1})c$. Q.E.D.

Theorem I is in complete analogy with theorem I of [2]. Theorem II of [2] also has an analogue here, and it can be proved along the same lines as in [2]. To simplify formulations, we use the concept of the logarithmic likelihood ratio statistics, defined by

$$\lambda_o = \log \sup_{H_o} \prod_{k=1}^{n} f(x_k,\underset{\sim}{\theta},\tau) - \log \sup_{\Omega} \prod_{k=1}^{n} f(x_k,\underset{\sim}{\theta},\tau),$$

(5)

$$\lambda_1 = \log \sup_{H_1} \prod_{k=1}^{n} f(x_k,\underset{\sim}{\theta},\tau) - \log \sup_{\Omega} \prod_{k=1}^{n} f(x_k,\underset{\sim}{\theta},\tau),$$

where $f(x,\underset{\sim}{\theta},\tau)$ is the density function, given in our case by (1). Substituting (1) in (5), we can express λ_o and λ_1 in terms of the sufficient statistics $S = \sum_{i=1}^{n} z_i(x_k)$ and $m = \max_{1\leq k\leq n} x_k$ as follows:

Denote by $\Omega(m)$ the intersection of Ω with the half-space $\{(\underset{\sim}{\theta},\tau) | \tau \geq m\}$; then we have

$$(6) \qquad \lambda_i = \sup_{H_i \cap \Omega(m)} (\Sigma\, \theta_i\, S_i - nb(\underset{\sim}{\theta},\tau)) - \sup_{\Omega(m)} (\Sigma\, \theta_i\, S_i - nb(\underset{\sim}{\theta},\tau))$$

for $i = 1, 2$.

Now consider the $p+2$ dimensional space with coordinates $n, S_1, S_2, \ldots , S_p, n\, m$. At each stage of sampling the point we reach in that space is a sufficient statistic, and the expected a posteriori loss when sampling is stopped there and H_i is decided is a function of that point, given by

$$(7) \qquad R_i(n,S_1, \ldots , S_p,n\, m) = R_i(n,\underset{\sim}{S},n\, m)$$

$$= \frac{\int_{H_i \cap \Omega(m)} e^{\underset{\sim}{\theta}\cdot\underset{\sim}{S} - nb(\underset{\sim}{\theta},\tau)} L(\underset{\sim}{\theta},\tau)\, dW(\underset{\sim}{\theta},\tau)}{\int_{\Omega(m)} e^{\underset{\sim}{\theta}\cdot\underset{\sim}{S} - nb(\underset{\sim}{\theta},\tau)} L(\underset{\sim}{\theta},\tau)\, dW(\underset{\sim}{\theta},\tau)} .$$

As in [2], we define the region $C(r)$ by

$$(8) \qquad C(r) = \{(n,\underset{\sim}{S},nm) \mid R_i \geq r ,\ i = 0,1\} ,$$

and proceed to find the intersection of $C(r)$ with the straight line $n = t,\ S_1 = k_1 t,\ S_2 = k_2 t,\ \ldots ,\ S_p = k_p t, nm = k_{p+1} t$, which yields

(9) $$r \leq \frac{\int_{H_i \cap \Omega(k_{p+1})} [e^{\underset{\sim}{\theta}\cdot\underset{\sim}{k} - b(\underset{\sim}{\theta},\tau)}]^n \, L(\underset{\sim}{\theta},\tau)\, dW}{\int_{\Omega(k_{p+1})} [e^{\underset{\sim}{\theta}\cdot\underset{\sim}{k} - b(\underset{\sim}{\theta},\tau)}]^n \, L(\underset{\sim}{\theta},\tau)\, dW}, \quad i = 0,1 .$$

For fixed k_{p+1} the regions of integration are now fixed, and the proof of theorem II of [2] applies, yielding the following

Theorem II: When the region $C(r)$ is transformed homothetically by a factor of $1/\log r^{-1}$, the transformed region approaches the limiting region B_o, given by

(10) $$B_o = \{(n,\underset{\sim}{S},nm) \mid \ell_i \geq -1,\ i = 0,1\} .$$

Combining theorems I and II, we can now prove Theorem III:

Theorem III: When the Bayes sampling region $B(c)$ is transformed homothetically by a factor of $1/\log c^{-1}$, the transformed region approaches the limiting region B_o, defined above.

Proof: First, theorem I implies $B(c) \supset C(2c + \frac{c}{\Delta} \log \frac{1}{c})$. Second, the fact throughout the complement of $C(c)$, stopping and deciding on the terminal action with the lower a posteriori risk leads to a loss that does not exceed the cost of a single observation, implies that no point in the complement of $C(c)$ can be in $B(c)$, and hence $C(c) \supset B(c)$. Now we have an inclusion from both sides:

(11) $$C(c) \supset B(c) \supset C(2c + \frac{c}{\Delta} \log \frac{1}{c}) .$$

Transforming all regions by $1/\log c^{-1}$, we obtain

$$(12) \qquad \frac{C(c)}{\log \frac{1}{c}} \supset \frac{B(c)}{\log \frac{1}{c}} \supset \frac{C(2c + \frac{c}{\Delta} \log \frac{1}{c})}{\log \frac{1}{c}} = \frac{C(2c + \frac{c}{\Delta} \log \frac{1}{c})}{\log \frac{1}{2c + \frac{c}{\Delta} \log \frac{1}{c}} + O(\log \frac{1}{c})} .$$

Applying theorem II to the left term with $r = c$ and to the right term with $r = 2c + (c/\Delta) \log (1/c)$, we see that both regions approach B_o , and therefore, so does $B(c)/\log c^{-1}$. Q.E.D.

4. Two Examples.

A) Let X be uniformly distributed on the interval $[0,\tau]$. We wish to test H_o: $0 < \tau \leq \tau_o$ against H_1: $\tau \geq \tau_1$. No exponential parameters θ_i are involved. The function $b(\underset{\sim}{\theta},\tau)$ can be chosen as $b(\tau) = \log \tau$. The logarithmic likelihood ratio statistics are given by

$$(13) \qquad \begin{cases} \lambda_o = 0 & \text{for} \quad m \leq \tau_o \\ \lambda_o = -\infty & \text{for} \quad m > \tau_o \end{cases}$$

and

$$(14) \qquad \begin{cases} \lambda_1 = 0 & \text{for} \quad m \geq \tau_1 \\ \lambda_1 = n \log (m/\tau_1) & \text{for} \quad m < \tau_1 \end{cases} .$$

Hence $\lambda_o \geq -1$ whenever $m \leq \tau_o$

and $\lambda_1 \geq -1$ whenever $m \geq \tau_1 e^{-1/n}$; the region B_o consists therefore of all points in the (n,mn)-plane at which

(15) $$\tau_1 e^{-\frac{1}{n}} \leq m \leq \tau_o .$$

The region $B_o \log \frac{1}{c}$, that may be used as an approximation to the Bayes region when c is small, is obtained by replacing n by $n/\log \frac{1}{c}$:

(16) $$\tau_1 c^{1/n} \leq m \leq \tau_o .$$

B. Let X be exponentially distributed on the half-line $(-\infty, \tau]$ with parameter θ . The density is

(17) $$f(x, \theta, \tau) = \theta e^{\theta(x-\tau)}$$

and we have $b(\theta, \tau) = \theta \tau - \log \theta$. We test H_o: $\tau \leq \tau_o$ against H_1: $\tau \geq \tau_1$.

The condition of a lower bound on $b(\theta, \tau) - b(\theta, \tau_o)$ is not fulfilled unless we restrict the parameter space to

(18) $$\Omega = \{(\theta, \tau) | \theta \geq K > 0\} .$$

The logarithmic likelihood ratio statistics are then given as follows:

(19) $$\begin{cases} \lambda_o = 0 & \text{when} \quad m \leq \tau_o \\ \lambda_o = -\infty & \text{when} \quad m > \tau_o \end{cases}$$

and, denoting the sample mean S/n by $\bar{x}$,

$$
(20) \quad \begin{cases}
\lambda_1 = 0 & \text{when } m \geq \tau_1 ; \\
\lambda_1 = -nK(\tau_1 - m) & \text{when } m < \tau_1 , \ \bar{x} \leq m - \frac{1}{K} \\
\lambda_1 = n - nK(\tau_1 - m) + n \log[K(m-\bar{x})] & \text{when } m < \tau_1 , \ m - \frac{1}{K} \leq \bar{x} \leq \tau_1 - \frac{1}{K} \\
\lambda_1 = -n \log \frac{\tau_1 - \bar{x}}{m - \bar{x}} & \text{when } m < \tau_1 , \ \bar{x} \geq \tau_1 - \frac{1}{K}
\end{cases}
$$

and B_o is the intersection of the region $\{(n,S,nm) \mid m \leq \tau_o\}$ with the region $\{(n,S,nm) \mid \lambda_1 \geq -1\}$.

The second region is rather complicated. However, if K is very small, we may assume that $\bar{x}$ is greater than $\tau_1 - \frac{1}{K}$ with high probability, and B_o is approximately

$$
(21) \qquad \{(n,S,nm) \mid \bar{x} + (\tau_1 - \bar{x}) \, e^{-1/n} \leq m \leq \tau_o\} .
$$

Correspondingly, the region $B_o \log c^{-1}$ is given by

$$
(22) \qquad \{(n,S,nm) \mid \bar{x} + (\tau_1 - \bar{x}) \, c^{1/n} \leq m \leq \tau_o\} .
$$

5. Multiple Truncation.

By various modifications of the method used here and in [2], some cases that are not formally included in section 3 may be solved. As a typical example we have chosen the testing of the midrange of a uniform distribution with both end points unknown.

This example, having two truncation parameters and no other parameters, is a special case of a random variable X whose density depends on $q+t$ functions of X $(f_1(X), \dots , f_q(x) ; g_1(X), \dots , g_t(x))$ in the following way:

$$(23) \qquad f(x, \theta_1, \dots , \theta_q ; \tau_1, \dots , \tau_t) = e^{\Sigma \theta_i f_i(x) - b(\underset{\sim}{\theta},\underset{\sim}{\tau})}$$

throughout the region $\{x \mid g_i \leq \tau_i , 1 \leq i \leq t\}$,

and $f(x,\theta_1, \dots , \theta_q ; \tau_1, \dots , \tau_t) = 0$ otherwise.

For all those random variables, theorem II can be stated and proved in terms of the sufficient statistic

$$(24) \quad (n, \sum_i f_1(X_i), \dots , \sum_i f_q(X_i) ; n \max_i g_1(X_i), \dots , n \max_i g_t(X_i)) .$$

Theorem I, however, has to be proved separately for each case.

In our example $q = 0$, $k = 2$, and the functions $f_i(X)$ are $f_1(X) = X$, $f_2(X) = -X$. Denoting the interval on which X is uniformly distributed by $[a,b]$, the truncation parameters are $\tau_1 = b$, $\tau_2 = -a$, and the midpoint $\frac{a+b}{2}$ is $\frac{\tau_1-\tau_2}{2}$.

We are testing H_o: $\frac{\tau_1-\tau_2}{2} \leq d_o$ against H_1: $\frac{\tau_1-\tau_2}{2} \geq d_1$. The requirement of a non degenerate indifference region can be satisfied by putting a bound on the range $(b-a)$. We therefore define the parameter space as

$$(25) \qquad \Omega = \{(\tau_1,\tau_2) \mid 0 < \tau_1 + \tau_2 \le K\} \ . \quad K > d_1 - d_o \ .$$

As Theorem I does not apply directly here, a similar result must be proved for this case. Again we define a fixed sample size procedure $\mathcal{J}(r)$. The procedure depends solely on the sample midrange

$$(26) \qquad D_n = \frac{1}{2} \left(\max_{1 \le i \le n} X_i + \min_{1 \le i \le n} X_i \right) ,$$

and is defined as follows:

"Take N observations, where N is the smallest integer greater than $\log r^{-1}/\log \frac{2K}{K-(d_1-d_o)}$ and decide "H_o" if $D_n \le \frac{1}{2}(d_o + d_1)$, and "H_1" otherwise."

By an elementary calculation, the distribution of the midrange is given by

$$(28) \qquad \begin{cases} \text{Prob}\{D_n \le d\} = 2^{n-1} \left(\frac{d-a}{b-a} \right)^n & \text{when } a \le d \le \frac{a+b}{2} , \\ \text{Prob}\{D_n \le d\} = 1-2^{n-1} \left(\frac{b-d}{b-a} \right)^n & \text{when } \frac{a+b}{2} \le d \le b . \end{cases}$$

To obtain a bound on the probability that $\mathcal{J}(r)$ will lead to saying H_o , when H_1 is true, we put $d = \frac{1}{2}(d_o + d_1)$ in (28), and apply to a and b the bounds $b-a \le K$, $a+b \ge 2d$, that hold throughout H_1 . We obtain the bound

(29) $\text{Prob}_{H_1}\{\mathcal{J}(r) \text{ leads to } "H_o"\} =$

$$\text{Prob}_{H_1}\{D_N \leq \tfrac{1}{2}(d_o + d_1)\} \leq 2^{N-1}\left(\frac{\frac{1}{2}(K - d_1 + d_o)}{K}\right)^N$$

$$= \frac{1}{2}\left(\frac{K - (d_1 - d_o)}{K}\right)^N < c \text{ .}$$

By symmetry, the same bound holds for the probability of an error of the first kind. Denoting $\log(2K/K-d_1+d_o)$ by Δ, we can now proceed as in section 3 and prove

(30) $$C(c) \supset B(c) \supset C(c + \frac{c}{\Delta}\log\frac{1}{c}) \text{ .}$$

Hence, the asymptotic formula obtained in section 3 is valid in the case we are treating here. To obtain B_o explicitly, we again consider the logarithmic likelihood statistics.

They can be expressed in terms of the sample midrange D and the sample range T by the following formulae:

(31) $$\begin{cases} \lambda_o = 0 & \text{when} \quad D \leq d_o \\ \lambda_o = -n\log(1+2\,\frac{D-d_o}{T}) & \text{when} \quad d_o < D \leq d_o + \frac{K-T}{2} \\ \lambda_o = -\infty & \text{when} \quad D > d_o + \frac{K-T}{2} \end{cases}$$

and

$$(32)\quad \begin{cases} \lambda_1 = 0 & \text{when} \quad D \geq d_1 \\ \lambda_1 = -n \log(1+2 \frac{d_1-D}{T}) & \text{when} \quad d_1 - \frac{K-T}{2} \leq D < d_1 \\ \lambda_1 = -\infty & \text{when} \quad D < d_1 - \frac{K-T}{2} \end{cases} .$$

The asymptotic shape B_o can now be expressed in terms of the coordinates (n,nD,nT) as follows:

$$(33)\quad B_o = \{(n,nD,nT) | d_1 - \tfrac{1}{2} \min[K-T,(e^{1/n}-1)] \leq D \leq d_o + \tfrac{1}{2} \min[K-T,(e^{1/n}-1)T]\} .$$

Again, the region $\log c^{-1} B_o$, that is the large-sample approximation to the Bayes sampling region, is obtained by replacing $e^{1/n}$ in (33) by $c^{-1/n}$.

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