APPLIED MATHEMATICS AND STATISTICS LABORATORIES
STANFORD UNIVERSITY
CALIFORNIA

MARKOV RENEWAL PROCESSES OF ZERO ORDER AND THEIR APPLICATION TO COUNTER THEORY

By
RONALD PYKE

TECHNICAL REPORT NO. 64

November 21, 1960

PREPARED UNDER CONTRACT Nonr-225(52)
(NR-342-022)
FOR
OFFICE OF NAVAL RESEARCH

MARKOV RENEWAL PROCESSES OF ZERO ORDER AND THEIR APPLICATION TO COUNTER THEORY

by

Ronald Pyke

TECHNICAL REPORT NO. 64

November 21, 1960

PREPARED UNDER CONTRACT Nonr-225(52)

(NR 342-022)

OFFICE OF NAVAL RESEARCH

This work was sponsored by the Army, Navy, and Air Force through the Joint Services Advisory Committee for Research Groups in Applied Mathematics and Statistics by Contract No. Nonr-225(52)(NR 342-022).

APPLIED MATHEMATICS AND STATISTICS LABORATORIES

STANFORD UNIVERSITY

STANFORD, CALIFORNIA

MARKOV RENEWAL PROCESSES OF ZERO ORDER AND THEIR APPLICATION TO COUNTER THEORY

by

Ronald Pyke *

1. Summary. A special class of Markov Renewal processes are studied in which the embedded Markov Chains are of zero-order. The distribution theory both exact and asymptotic of the number of visits to a state is studied. In Section 6 a special case arising in counter theory is described and the corresponding results stated. The author's interest in more general Markov Renewal processes and this special case was motivated by the problem encountered where impulses emitted from a radioactive source are to be detected by a Multiple Channel Analyzer.

2. Introduction and Notation: In the present state of the probabilistic theories of queueing, inventories, counters, dams, etc., a principal role is played by the theory of Renewal processes. A Renewal process is defined simply as a sequence of mutually independent and identically distributed random variables (r.v.'s), or equivalently, of their partial sums and are used to represent successive interarrival times of customers at a queue, demands and supplies at a store, water inputs at a dam, and so on. The study of these processes, Renewal Theory studies primarily exact and asymptotic properties of the random number of sums which do not exceed a fixed value. The reader is referred to

the extensive exposition of Renewal Theory by Smith [1] and, for its relationship to Inventory Theory, to Karlin (Chap. 15 of [2]).

In this paper, a generalization of Renewal Theory is suggested which studies sequences of r.v.'s that are independent but not necessarily identically distributed. It may best be described as follows. Demands are made on an inventory at random time points determined by a Renewal process. These demands may be made for any one of a finite number of types of goods. Given the type of good demanded, the store requires a random time in which to supply the customer, during which time no new demands are accepted. Hence the distribution function (d.f.) of the time until the next accepted demand depends on the type of goods previously demanded. A particular application of this model to a counter problem is given below in section 6.

To be precise, let $\{J_n : n \geq 0\}$ be an integer-valued Renewal Process having the d.f. determined by

$$(2.1) \qquad \Pr[J_n = k] = p_k > 0 \qquad (k = 1, 2, \ldots, m < \infty)$$

for all $n \geq 0$. Let F_i $(i = 1, \ldots, m)$ be d.f.'s satisfying $F_i(0) = 0$ for each i. Then define the stochastic process $\{X_n : n \geq 1\}$ by specifying $X_o = 0$ and

$$(2.2) \qquad \Pr[X_n \leq x | X_o, X_1, \ldots, X_{n-1}, J_o, J_1, \ldots, J_n] = F_{J_{n-1}}(x) \text{ a.s.}$$

for all $x > 0$ and $n > o$. The interpretation of these quantities in terms of the foregoing description is as follows. m is the number of different types of goods available, J_n is the type demanded by the n-th customer, p_k is the probability that the n-th customer selects the k-th type, F_k is the d.f. of the time between two consecutive accepted demands when the previous demand was for type k, and X_n is the time between the arrivals of the $(n-1)$-th and n-th accepted demands.

Set $S_n = X_o + X_1 + \cdots + X_n$, and define

$$N(t) = \sup\{k: S_k \leq t\}$$

and

$$N_j(t) = \text{no. if times } J_n = j \quad \text{for} \quad 0 < n \leq N(t)$$

for $t \geq 0$ and $j = 1, \ldots, m$. Clearly $N(t) = N_1(t) + \cdots + N_m(t)$. Furthermore, define

$$(2.3) \qquad Q_j(t) = p_j F_j(t), \qquad H(t) = \sum_{j=1}^{m} Q_j(t)$$

An immediate consequence of (2.2) is

Lemma 2.1: The process $\{X_n : n \geq 1\}$ is Renewal process with common d.f.; H, and hence the expectation of $N(t)$, is the Renewal function associated with H.

In fact, when $m = 1$, nothing more than a Renewal process has been defined by the above.

The main object of this paper is to study the exact and asymptotic properties of the $N_j(t)$ $(j = 1, \ldots, m)$.

If one compares the above definitions with those used by the author in [3] to define a Markov Renewal process (M.R.P.), one sees that the kind of process defined above is but a special case of an M.R.P. Retaining the same notation, the most general M.R.P. is determined by its matrix of transition distributions

Q_{ij} $(i,j = 1,2, \ldots, m)$, where for all x

(2.4) $$Q_{ij}(x) = \Pr[J_n = j, X_n \leq x | J_o, J_1, \ldots, J_{n-1} = i, X_o, X_1, \ldots, X_{n-1}] .$$

These processes were first studied by Takács [4] in 1954, and have since been studied by many others. References may be found, for example, in [5]. It follows from (2.1) and (2.2) that for the processes considered in this paper the Q_{ij}'s of (2.4) are given by $Q_{ij}(x) = p_j F_i(x)$.

This factorization will permit more explicit results to be obtained then were possible in [3] and [5] for the general case, as well as simpler derivations of some of the results in [5].

A Markov Chain is said to be of zero-order if the transition probabilities p_{ij} are functions of j only; i.e. the transition matrix has equal rows. (With this terminology, the Renewal process $\{J_n : n \geq 0\}$ defined by (2.1) is equivalently a zero-order Markov Chain.) Analogously, therefore, define a zero-order M.R.P. to be one for which the Q_{ij}'s do not depend on i. Such a process would be obtained by replacing (2.2) with

$$(2.5) \qquad \Pr[X_n \leq x | X_0, X_1, \ldots, X_{n-1}, J_0, J_1, \ldots, J_n] = F_{J_n}(x) \quad \text{a.s.}$$

in which case $Q_{ij}(x) = p_j F_j(x) = Q_j(x)$ as required. The zero-order M.R.P. does not seem to be directly applicable to most problems, since, for example, it would mean that the interarrival time of the (n-1)-th and n-th customer would depend on the type of goods demanded by the n-th (rather than the (n-1)-th) customer. However, the zero-order M.R.P. and the processes studied in this paper, are essentially equivalent probabilistically, since the two-dimensional process $\{J_n, X_n) : n \geq 1\}$ associated with the former, and $\{(J_{n-1}, X_n) : n \geq 1\}$ with the latter have the same probabilistic structure. This relationship is studied further in Section 5 below, and justifies the title of this paper.

Set

$$b_j = \int_0^\infty x \, dF_j(x), \qquad \eta = \int_0^\infty x \, dH(x) = \sum_{j=1}^{m} p_j b_j$$

and assume that these moments are finite. The Laplace-Stieltjes transforms of a function will, whenever it exists, be denoted by the corresponding lower case letter. Unless otherwise stated, the index j will run from 1 to m through the integers, and s and t will always denote non-negative quantities.

3. <u>Distribution theory for</u> $N_j(t)$: For all $1 \leq i, j \leq m$, $k \geq 0$, set

$$v_{ij}(k;t) = \Pr[N_j(t) = k | J_0 = i]$$

$$G_{ij}(t) = \Pr[N_j(t) > 0 | J_0 = i]$$

It should first be emphasized that if one knew the first-passage-time d.f.'s G_{ij} for each i, then most of the properties of $N_j(t)$ could be obtained directly from known results of Renewal Theory. For example, it is known, and easily proven, that

(3.1) $$v_{ij}(0;\ t) = 1 - G_{ij}(t)$$

(3.2) $$v_{ij}(k;\ t) = G_{ij}(t) * G_{jj}^{*(k-1)}(t) * [1 - G_{jj}(t)] \qquad (k > 0)$$

where $*$ denotes Laplace-Stieltjes (LS) convolution and for any function F of bounded variation F^{*0} denotes the d.f. degenerate at zero, $F^{*n} = F * F^{*(n-1)}$ for $n > 0$ and $(1-F)^{*(-1)} = \sum_{n=0}^{\infty} F^{*n}$ whenever these operations exist. Furthermore, it follows directly from Renewal Theory that $M_j(t) \equiv E[N_j(t)]$ is given in terms of the G_{ij} by

(3.3) $$M_j(t) = \sum_{i=1}^{m} p_i G_{ij}(t) * [1 - G_{jj}(t)]^{*(-1)}$$

and that as $t \to +\infty$

(3.4) $$t^{-1} N_j(t) \to \mu_j^{-1} (\text{a.s.}),$$

$$M_j(t) = t/\mu_j + \mu_j^{(2)}/2\mu_j^2 - \bar{\mu}_j/\mu_j + o(1)$$

where μ_j and $\mu_j^{(2)}$ are the first and second moments of G_{jj} and $\bar{\mu}_j$ is the first moment of the d.f. $\sum_{i=1}^{m} p_i G_i$.1/ To see that the second expression of (3.4) is indeed a consequence of Renewal Theory, recall (cf. (1.4) of [1]) that since $[1 - G_{jj}(t)]^{*(-1)}$ is simply the Renewal function associated with the d.f. G_{jj},

$$[1 - G_{jj}(t)]^{*(-1)} = t/\mu_j + \mu_j^{(2)}/2\mu_j^2 + o(1)$$

and therefore one obtains

$$M_j(t) - t/\mu_j + \mu_j^{(2)}/2\mu_j^2 = - [1 - \Sigma p_i G_{ij}(t)] * [1 - G_{jj}(t)]^{*(-1)} + o(1)$$

However by the Key Renewal Theorem, (1.3) of [1], the right hand side of this last expression converges to $-\bar{\mu}_j/\mu_j$ as $t \to +\infty$ which gives (3.4). The asymptotic behaviour of the variance of $N_j(t)$ may be derived from known results in an analogous way. For clearly

$$E\{[N_j(t)]^2 | J_o = i\} = G_{ij}(t) * \sum_{k=o}^{\infty} (2k + 1)G_{jj}^{*k}(t)$$

It is known (cf. (7.6) and (7.9) of [6]) that

$$\sum_{k=0}^{\infty} (2k + 1)G_{jj}^{*k}(t) = t^2/\mu_j{}^2 + t(2\mu_j^{(2)} - \mu_j^2)/\mu_j^3 + o(t)$$

and hence

$$E\{[N_j(t)]^2\} = t^2\mu_j^{-2} + t(2\mu_j^{(2)} - \mu_j^2 - 2\bar{\mu}_j\mu_j)\mu_j^{-3} + o(t)$$

This in conjunction with (3.4) yields

(3.5) $$\text{var}\{N_j(t)\} = t\sigma_j^2\mu_j^{-3} + o(t)$$

where $\sigma_j^2 = \mu_j^{(2)} - \mu_j^2$. Moreover, it also follows from Renewal Theory that $[N_j(t) - t/\mu_j]t^{-1/2}$ is asymptotically Normal with mean zero and and variance $\sigma_j^2 \mu_j^{-3}$.

The above discussion is based on the supposition that the functions G_{ij} were known. It is clear from (3.1) and (3.2) that the G_{ij}'s may be determined from the v_{ij}'s and vice versa. Since (3.1) is the simpler of the two relations, the v_{ij}'s are obtained first. Set $K_j(t) = H(t) - Q_j(t) = \Pr[X_n \leq t, J_{n-1} \neq j]$. Then since the event $[N_j(t) = k]$ is characterized by the occurence of either $k - 1$ events $[J_n = j]$ and $N(t) - k$ (≥ 0) events for $n = 1, 2, \ldots, N(t) - 1$ together with $[J_{N(t)} = j]$, or k events $[J_n = j]$ and $N(t) - k - 1$ (≥ 0) events $[J_n \neq j]$ for $n = 1, 2, \ldots, N(t) - 1$ together with $[J_{N(t)} \neq j]$, one is able to prove

Lemma 3.1: For $k > 0$

$$(3.6) \quad v_{ij}(k; t) = p_j F_i(t) * Q_j^{*(k-1)}(t) * [1 - K_j(t)]^{*(-k-1)} * [1 - H(t)]$$

$$(3.7) \quad v_{ij}(0; t) = 1 - p_j F_i(t) * [1 - K_j(t)]^{*(-1)}$$

Proof: By the remark preceeding this lemma, one obtains for $k > 0$

$$v_{ij}(k;t) = F_i(t) * \sum_{n=k}^{\infty} \{ \binom{n-1}{k-1} p_j \, Q_j^{*(k-1)}(t) * K_j^{*(n-k)}(t) * [1 - F_j(t)]$$
$$+ \binom{n-1}{k} Q_j^{*k}(t) * K_j^{*(n-k-1)}(t) * [1 - p_j - K_j(t)]\}.$$

where $\binom{k-1}{k} = 0$. (3.6) is just an algebraic simplification of this latter expression, recalling that $K_j = Q_j = H_j$. (3.7) may either be obtained in a similar manner or simply from $1 = \sum_{k=0}^{\infty} v_{ij}(k;t)$. Q.e.d.

By (3.1), the first-passage-time d.f.'s are obtained from the above lemma, as given in

Lemma 3.2: For $t \geq 0$, $s \geq 0$

$$(3.8) \qquad G_{ij}(t) = p_j F_i(t) * [1 - K_j(t)]^{*(-1)}, \quad g_{ij}(s) = \frac{p_j f_i(s)}{1-k_j(s)}$$

Upon introducing the generating functions

$$\varphi_{ij}(z;t) = \sum_{k=0}^{\infty} z^k v_{ij}(k;t), \qquad \psi_{ij}(z;s) = \int_0^{\infty} e^{-st} d_t \varphi_{ij}(z;t)$$

taking $\varphi_{ij}(z;0-) = 0$ one may express Lemma 3.1 equivalently as

<u>Lemma 3.3</u>: For $|z| \leq 1, \; s \geq 0$

$$(3.9) \qquad \varphi_{ij}(z;t) = 1 - (1-z)p_j F_i(t) * [1-H(t) + (1-z)Q_j(t)]^{*(-1)}$$

$$(3.10) \qquad \psi_{ij}(z;s) = 1 - \frac{(1-z)p_j f_i(s)}{[1-h(s)+(1-z)q_j(s)]}$$

The moments of $N_j(t)$ may be computed from (3.9). In particular, setting $\varphi_j(z;t) = \sum_{i=1}^{m} p_i \varphi_{ij}(z;t)$ as the unconditional generating function of $N_j(t)$, one has

$$\frac{\partial}{\partial z} \varphi_j(z;t) = p_j[1-H(t)] * H(t) * [1-H(t) + (1-z)Q_j(t)]^{*(-2)}$$

from which it follows that

$$(3.11) \qquad M_j(t) = p_j H(t) * [1-H(t)]^{*(-1)}, \; m_j(s) = p_j h(s)[1-h(s)]^{-1}$$

This result is interesting in that upon setting $M(t) = E\{N(t)\}$, Lemma 2.1 implies that $M_j(t) = p_j M(t)$. From (3.8), the moments of the first-passage-times may be obtained, giving in particular

$$(3.12) \qquad \mu_j = \eta/p_j, \quad \mu_j^{(2)} = \eta^{(2)}/p_j + 2(\eta/p_j - b_j)\eta/p_j$$

where $\eta^{(2)} = \int_0^\infty x^2 \, dH(x)$. Consequently, setting $\sigma^2 = \eta^{(2)} - \eta^2$,

$$(3.13) \qquad \sigma_j^2 = \sigma^2/p_j + \eta(\eta - 2b_j + \eta/p_j)/p_j$$

A striking feature of (3.12) is the fact that as j varies, the variances σ_j^2 differ only through the effect of the means and the probabilities p_j. The second moments of the d.f.'s F_j, except as they occur in σ^2, do not appear.

The two moments μ_j and σ_j^2 may also be obtained in a different manner. From (3.11) one obtains

$$\lim_{s \to o} sm_j(s) = \lim_{s \to o} p_j s[1 - h(s)]^{-1} = p_j/\eta$$

and

$$\lim_{s \to o} [m_j(s) - p_j/\eta s] = p_j \eta^{-2} \lim_{s \to o} \{s^{-1}\eta[h(s) - 1] + \eta^{(2)}/2\}.$$

$$= p_j(\eta^{(2)}/2 - \eta^2)\ \eta^{-2}$$

However, (3.4) implies that

$$\lim_{s \to o} sm_j(s) = \mu_j^{-1}, \qquad \lim_{s \to o} [m_j(s) - 1/\mu_j s] = \mu_j^{(2)}/2\mu_j^{\ 2} - \bar{\mu}_j/\mu_j$$

It is easily shown that the solutions of these equations check with (3.12) and (3.13) since $\bar{\mu}_j = \eta + \eta/p_j - b_j$.

In the sentence following (3.5) it is pointed out that $N_j(t)$ is asymptotically normal. One must expect that, furthermore, the vector $(N_1(t), N_2(t), \ldots , N_m(t))$ has a limiting joint normal distribution. This fact is contained in a central limit theorem for a wide class of functionals of the general M.R.P. which is proved in [7], the limiting covariance matrices also being determined. However, for the processes being studied in this paper, a simpler approach is possible. It hinges on the fact that in the present paper $\{(J_{n-1}, X_n): n \geq 1\}$ is a sequence of independent and identically distributed random vectors. Consider any two-dimensional function f, set $f(J_{n-1}, X_n) = f_n$ and write $W_f(t) = \sum_{n=1}^{N(t)} f_n$. Let $A = E(f_n) < \infty$. Then

$$(3.14) \qquad t^{-1/2}[W_f(t) - tA/\eta]$$

$$= \frac{\sum_{n=1}^{N(t)} (f_n - x_n A/\eta)}{[N(t)]^{1/2}} \cdot [N(t)/t]^{1/2} + [t - S_{N(t)}]A/\eta t^{1/2}$$

If $\eta < \infty$, known results of Renewal Theory give $N(t)/t \to \eta^{-1}$(a.s.) and $t - S_{N(t)}$ converging in law. Moreover, a straight forward argument (cf. [8]) shows that the first factor on the right side of (3.14) has the same limiting d.f. at $t \to \infty$ as does $\sum_{n=1}^{k} (f_n - X_n A/\eta)\, k^{-1/2}$ as $k \to \infty$, namely normal with mean zero and variance that of $f_n - X_n A/\eta$. Therefore one obtains

<u>Lemma 3.4</u>: If the variance of $f_n - X_n A/\eta$, B say, exists and is finite, then $W_f(t)$ is asymptotically normal with mean $tA\eta^{-1}$ and variance $tB\eta^{-1}$.

As a corollary to this lemma one obtains the asymptotic joint normality of $(N_1(t), \ldots, N_m(t))$ by defining f to satisfy $f(k, x) = u_k$ for arbitrary numbers u_k. If Σ, with elements σ_{ij}, denotes the covariance matrix for this limiting distribution, then the diagonal elements $\sigma_{jj} = \sigma_j^2$ have been determined above in (3.13) while the off-diagonal elements σ_{ij} $(i \neq j)$ are determined as follows. Let $f(k, x) = 1$ or 0 according as k is or is not equal to either i or j, in which case $W_f(t) = N_i(t) + N_j(t)$. By Lemma 3.4 this sum has asymptotic variance $tB\eta^{-1}$, where B is easily computed to be

$$B = (p_i + p_j)[1 - \frac{2}{\eta}(p_i b_i + p_j b_j)] + (p_i + p_j)^2 (\sigma^2 \eta^{-2} + 1).$$

Upon subtracting from this the asymptotic variances of $N_i(t)$ and $N_j(t)$ as given in (3.13) and dividing by 2 one obtains

$$\sigma_{ij} = \frac{p_i p_j}{\eta}[\sigma^2 \eta^{-2} - (b_i + b_j)/\eta + 1] \qquad (i \neq j)$$

4. <u>The process $Z(t)$</u>: Define $Z_t = J_{N(t)}$. Then the process $\{Z_t : t \geq 0\}$ indicates at each moment t the state of the system. As is the case for Markov processes, one is interested in the one-dimensional d.f.'s of this process under all initial conditions. For this reason, define $P_{ij}(t) = \Pr[Z_t = j | Z_o = i]$ and $\pi_{ij}(s) = \int_o^\infty e^{-st}\, dP_{ij}(t)$. Then as an application of Theorem 3.1 of [5] together with Lemma 3.2 above, one obtains

Lemma 4.1: For $i \neq j$

$$P_{jj}(t) = [1 - F_j(t)] * [1 - K_j(t)] * [1 - H(t)]^{*(-1)},$$

$$\pi_{jj}(s) = \frac{[1 - f_j(s)][1 - k_j(s)]}{1 - h(s)}$$

$$P_{ij}(t) = p_j F_i(t) * [1 - F_j(t)] * [1 - H(t)]^{*(-1)},$$

$$\pi_{ij}(s) = \frac{p_j f_i(s)[1 - f_j(s)]}{1 - h(s)}$$

The question of limiting stationarity of the process $\{Z_t : t \geq 0\}$ and of the original process is also important and may be described completely from results in [5]. By limiting stationarity, one refers to the existence of a limit as $t \to \infty$ of the probabilities

$$R_{jk}^{(i)}(x; t) = \Pr[Z_t = j, J_{N(t)+1} = k, S_{N(t)+1} \leq t + x | Z_o = i]$$

It may easily be shown (cf. (7.8) of [5]) that

$$R_{jk}^{(i)}(x; t) = p_k[F_j(t + x) - F_j(t)] * G_{ij}(t) * [1 - G_{jj}(t)]^{*(-1)}$$

$$+ \delta_{ij}[F_j(t + x) - F_j(t)]$$

It can then be deduced from Renewal Theory (cf. Th 7.2 of [5]) that

$$\lim_{t \to \infty} R_{jk}^{(i)}(x; t) = p_k \mu_j^{-1} \int_o^x [1 - F_j(y)]dy$$

$$= p_j p_k \eta^{-1} \int_o^x [1 - F_j(y)]dy. \tag{4.1}$$

the last line following from (3.12). Not only does one have this limiting stationarity, but by changing the d.f. of (J_o, X_1) to that given by

$$\Pr[J_o = j, x_1 \le x] = \lim_{t \to \infty} R_{jk}^{(i)}(x; t) = p_j p_k \eta^{-1} \int_o^x [1 - F_j(y)]dy$$

the resulting process is strictly stationary (cf. Th. 7.3 of [5]).

5. Zero-order M.R.P.: In section 2 a zero-order M.R.P. was defined as one for which the function Q_{ij} of (2.4) satisfies $Q_{ij} = Q_j = p_j F_j$. Such a process is defined exactly as those studied above, for which $Q_{ij} = p_j F_i$, but with the defining relation (2.5) replacing that in (2.2). These two processes are very closely related probabilistically as mentioned at the end of section 2. Moreover, one can show that in a natural way these two types of M.R.P's are the reverses in time of each other. For example, using a bar over a letter to denote that the indicated quantity is defined for the zero-order process, it is clear by reversing the processes that the recurrence time d.f.'s are the same for both processes. That is, $G_{jj} = \overline{G}_{jj}$.

Let it suffice to list the pertinent results for the zero-order processes, leaving their straightforward derivations to the reader. Lemma 2.1 remains unchanged.

$$\bar{v}_{ij}(k;\ t) = \bar{v}_j(k;\ t) = G_{jj}^{*k} * [1 - G_{jj}(t)] = v_{jj}(k;\ t)$$

$$\bar{M}_j(t) = G_{jj}(t) * [1 - G_{jj}(t)]^{*(-1)}$$

$$\bar{\Psi}_{ij}(z;\ s) = \bar{\Psi}_{jj}(z;\ s)$$

The asymptotic normality of $(\bar{N}_1(t), \ldots, \bar{N}_m(t))$ is as before, since one simply considers functions $f(\bar{J}_n, \bar{X}_n)$ instead of $f(J_{n-1}, X_n)$, noting that $f(\bar{J}_n, \bar{X}_n)$ and $f(\bar{J}_{n-1}, \bar{X}_n)$ are identically distributed. It is interesting to note that

$$\bar{P}_{jj}(t) = \bar{P}_{ij}(t) = 1 - K_j(t), \qquad \bar{\pi}_{jj}(s) = \bar{\pi}_{ij}(s) = 1 - k_j(s)$$

Furthermore

$$\bar{R}_{jk}^{(i)}\ (x;\ t) = p_k[F_k(t + x) - F_k(t)] * G_{jj}(t) * [1 - G_{jj}(t)]^{*(-1)}$$

$$+ \delta_{ij}[F_k(t + x) - F_k(t)]$$

and

$$\lim_{t \to \infty} \bar{R}_{jk}^{(i)}\ (x;\ t) = p_k p_j\ \eta^{-1} \int_0^x [1 - F_k(y)]dy$$

$$= \lim_{t \to \infty} R_{kj}^{(i)}\ (x;\ t)$$

as is expected because of the reversibility and the known equality of the "age" and "excess" limiting distributions of Renewal Theory.

6. Application to a counter problem: The particular application which motivated the present study, as well as the author's interest in the more general Markov Renewal processes, is the following counter problem encountered in 1957 at the U.S. Naval Radiological Laboratories in San Francisco.[2/] Impulses are emitted from a radioactive source. These impulses are detected by a Multiple Channel Analyser, which is simply a series of m Geiger-Müller counters (usually $m = 256$) together with a voltmeter, so constructed that the voltages of the incoming impulses are first measured to determine into which of m voltage intervals, or channels, they fall, and then are forwarded to the counter assigned to count the impulses that fall into that channel. Thus a histogram can be constructed which approximates the true frequency function (spectrum) of the voltages of the impulses emitted from the given radioactive material. If the half-life of this material is sufficiently long, then the interarrival times of the successive impulses are exponentially distributed with some parameter $\lambda > 0$, usually unknown. Let f be the theoretical spectrum of the given material and assume that there exists a finite constant $\alpha > 0$ such that

$$(6.1) \qquad \int_0^{m\alpha} f(y)\, dy = 1, \qquad p_j \equiv \int_{(j-1)\alpha}^{j\alpha} f(y)\, dy > 0$$

for $1 \leq j \leq m$. These assumptions are sufficiently general to cover most of the situations encountered, while only slight changes are required to adapt them to all other situations. The Geiger-Müller counters present in the Analyzer are Type-I counters (cf. [9]), which is to say that all impulses which arrive during the short period of time necessary for the counter to actually register a count, (the deadtime), are not detected. The deadtime assumption for the Analyser is slightly more in that no impulses can be detected by any counter while one counter is registering a count. Moreover, the deadtime is a constant, proportional to the voltage of the impulse being counted. Without too much loss of generality it is assumed that the deadtime arising from an impulse with voltage in the interval $((j-1)\alpha, j\alpha)$ is jc.

If one lets J_n denote the channel into which the n-th registered impulse falls, and X_n denote the elapsed time between the (n-1)-th and n-th registered impulse, then in the notation of section 2 one has that p_j is given in (6.1) and

$$F_j(x) = \max\{0, 1 - e^{-\lambda(x-cj)}\}, \quad f_j(s) = e^{-jcs}\lambda(\lambda+s)^{-1}$$

$$(6.2) \qquad b_j = cj + \lambda^{-1}, \qquad \eta = \mu_s + \lambda^{-1}$$

$$\sigma^2 = \sigma_s^2 + \lambda^{-2}$$

where

$$\mu_s = c\sum_{j=1}^{m} jp_j \quad \text{and} \quad \sigma_s^2 = c^2\Big[\sum_{j=1}^{m} j^2 p_j - \Big(\sum_{j=1}^{m} jp_j\Big)^2\Big]$$

are the mean and variance of the discretized spectrum, or density, f. In practice the actual mean and variance of f, when known, might well be substituted into these expressions.

Of particular importance to this counter problem is the asymptotic behavior of $N(t), N_1(t), \ldots, N_m(t)$ which may be obtained from the foregoing sections, by substitution of (6.2) into the appropriate formulae. In particular one obtains that as $t \to \infty$

$$(6.3) \qquad t^{-1} M_j(t) \sim 1/\mu_j = p_j(\mu_s + 1/\lambda)^{-1}$$

$$(6.4) \qquad t^{-1} \operatorname{var}\{N_j(t)\} \sim \sigma_j^2 \mu_j^{-3}$$

$$= \frac{p_j}{(\mu_s + 1/\lambda)^2} \left\{ \frac{\sigma_s^2 + \mu_s^2}{\mu_s + 1/\lambda} + (\mu_s + 1/\lambda)/p_j - 2cj \right\}$$

If the Multiple Channel Analyser were ideal, in the sense that all deadtimes were zero (i.e. $c = 0$), then the corresponding results for (6.3) and (6.4) are $p_j\lambda$ in both, with strict equality holding. Of course, for this case, the processes $N(t)$ and $N_j(t)$ are Poisson processes with parameters λ and λp_j respectively, a fact which may be checked in Lemma 3.3, in which $\psi_{1j}(z; s) = s[s + (1 - z)p_j\lambda]^{-1}$. Another quantity of interest in counter models is bias. The bias of a counter is defined as the limit of the ratio of the average number of counts $M_j(t)$ to the corresponding quantity for the ideal counter.

Thus

$$\text{Bias of channel } j = \lim_{t \to \infty} M_j(t)/\lambda p_j t = (1 + c\mu_s\lambda)^{-1}$$

$$= \lim_{t \to \infty} M(t)/\lambda t$$

$$= \text{Bias of Analyser.}$$

Further quantities of interest, including the asymptotic behavior of the numbers of counts in the several channel may also be derived from the theory of the foregoing sections.

Footnotes

*/ This work was completed while the author was at Columbia University.

1/ Here and in what follows it is tacitly being assumed that G_{jj} (or equivalently, at least one of the F_i) is a non-lattice d.f. The results which obtain when this is not the case are easily deduced. Cf. section 6 (c) of [5].

2/ The author is grateful to Miss Marion Sandomire for communicating the problem.

R E F E R E N C E S

[1] W.L. Smith, "Renewal Theory and its ramifications," J. Roy. Statist. Soc. B., Vol. 20 (1958) pp. 243-302.

[2] Arrow, Karlin and Scarf, <u>Studies in the mathematical theory of inventory and production</u>, Stanford University Press, 1958

[3] R. Pyke, "Markov Renewal processes: Definitions and preliminary properties," Technical Report No. 10 (1959) Contract Nonr-266, Project No. 042-205, Columbia University.

[4] L. Takács, "Some investigations concerning recurrent stochastic processes of a certain type," Magyar Tud. Akad. Mat. Kutató Int. Kozl. Vol. 3, 1954 pp. 115-128 (Hungarian)

[5] R. Pyke, "Markov Renewal Processes with finitely many states," Technical Report No. 11 (1959) Contract Nonr-266, Project No. 042-205, Columbia University.

[6] W.L. Smith, "Asymptotic Renewal theorems", Proc. Roy. Soc. Edinb. (A) Vol. 64, (1954) pp. 9-48.

[7] R. Pyke, "Limit theorems for Markov Renewal processes", to appear.

[8] D.G. Kendall, "A note on Doeblin's central limit theorem", Proc. A.M.S. Vol. 8, (1957) pp. 1037-1039.

[9] R. Pyke, "On Renewal processes related to Type I and Type II counter models," Ann. Math. Stat. Vol. 29 (1958) pp. 737-754.

STANFORD UNIVERSITY

TECHNICAL REPORT DISTRIBUTION LIST

CONTRACT Nonr-225(52)

(NR 342-022)

Armed Services Technical
Information Agency
Arlington Hall Station
Arlington 12, Virginia 10

Commanding Officer
Office of Naval Research
Branch Office
Navy No. 100, Fleet P.O.
New York, N.Y. 2

Commanding Officer
Office of Naval Research
Branch Office
1000 Geary Street
San Francisco 9, California 1

Commanding Officer
Diamond Ordnance Fuze Labs.
Washington 25, D.C. 1

Commanding Officer
Picatinny Arsenal (ORDBB-TH8)
Dover, New Jersey 1

Commanding Officer
Watertown Arsenal (OMRO)
Watertown 72, Massachusetts 1

Commanding Officer
Attn: W.A. Labs.
Watertown Arsenal
Watertown 72, Massachusetts 1

Commanding Officer
Watervliet Arsenal
Watervliet, New York 1

Commanding Officer
Attn: Inspection Division
Springfield Armory
Springfield, Massachusetts 1

Commanding Officer
Signal Corps Electronic Research
Unit, EDL
9560 Technical Service Unit
P.O. Box 205
Mountain View, California 1

Commanding Officer
9550 Technical Service Unit
Army Liaison Group, Project Michigan
Willow Run Research Center
Ypsilanti, Michigan 1

Commanding Officer
Engineering Research & Development Labs.
Fort Belvoir, Virginia 1

Commanding General
Frankford Arsenal (ORDBA-LC)
Bridesburg Station
Philadelphia 37, Pennsylvania 1

Commanding General
Rock Island Arsenal
Rock Island, Illinois 1

Commanding General
Redstone Arsenal (ORDDW-QC)
Huntsville, Alabama 1

Commanding General
White Sands Proving Ground
(ORDBS-TS-TIB)
Las Cruces, New Mexico 1

Commanding General
Attn: Paul C. Cox, Ord. Mission
White Sands Proving Ground
Las Cruces, New Mexico 1

Commanding General
Attn: Technical Documents Center
Signal Corps Engineering Laboratory
Fort Monmouth, New Jersey 1

Commanding General
Ordnance Weapons Command
Attn: Research Branch
Rock Island, Illinois 1

Commander
Wright Air Development Center
Attn: ARL Tech. Library, WCRR
Wright-Patterson Air Force Base, Ohio 1

Commander
Western Development Division, WDSIT
P.O. Box 262
Inglewood, California 1

Chief, Research Division
Office of Research & Development
Office of Chief of Staff
U.S. Army
Washington 25, D.C. 1

Chief, Computing Laboratory
Ballistic Research Laboratory
Aberdeen Proving Ground, Maryland 1

Director
National Security Agency
Attn: REMP-1
Fort George G. Meade, Maryland 1

Director of Operations
Operations Analysis Div., AFOOP
Hq., U.S. Air Force
Washington 25, D.C. 1

Director
Snow, Ice & Permafrost Research
Establishment
Corps of Engineers
1215 Washington Avenue
Wilmette, Illinois 1

Director
Lincoln Laboratory
Lexington, Massachusetts 1

Department of Mathematics
Michigan State College
East Lansing, Michigan 1

Department of Mathematics
University College
Rutgers University
New Brunswick, New Jersey 1

Document Library
U.S. Atomic Energy Commission
19th & Constitution Avenues N.W.
Washington 25, D.C. 1

Headquarters
Oklahoma City Air Materiel Area
United States Air Force
Tinker Air Force Base,
Oklahoma 1

Institute of Statistics
North Carolina State College of A & E
Raleigh, North Carolina 1

Jet Propulsion Laboratory
California Institute of Technology
Attn: A.J. Stosick
4800 Oak Grove Drive
Pasadena 3, California 1

Librarian
The RAND Corporation
1700 Main Street
Santa Monica, California 1

NACA
Attn: Mr. E.B. Jackson, Office of
Aero Intelligence
1724 F Street, N.W.
Washington 25, D.C. 1

National Applied Mathematics Labs.
National Bureau of Standards
Washington 25, D.C. 1

Naval Inspector of Ordnance
U.S. Naval Gun Factory
Washington 25, D.C.
Attn: Mrs. C.D. Hock 1

Office, Asst. Chief of Staff, G-4
Research Branch, R & D Division
Department of the Army
Washington 25, D.C. 1

Office of Technical Services
Department of Commerce
Washington 25, D.C. 1

Technical Information Officer
Naval Research Laboratory
Washington 25, D.C. 6

Technical Director
Combat Development Department
Army Electronic Proving Ground
Fort Huachuca, Arizona 1

Technical Information Service
Attn: Reference Branch
P.O. Box 62
Oak Ridge, Tennessee 1

Dr. Forman S. Acton
Department of Mathematics
Princeton University
Princeton, New Jersey 1

Mr. Irving B. Altman
Inspection & QC Division
Office, Asst. Secretary of Defense
Room 2B870, The Pentagon
Washington 25, D.C. 1

Professor T.W. Anderson
Department of Statistics
Columbia University
New York 27, New York 1

Dr. Merle M. Andrew
Chief, Mathematics Division
Air Force Office of
Scientific Research
Washington 25, D.C. 1

Professor Robert Bechhofer
Dept. of Industrial and Engineering
Administration
Sibley School of Mechanical Engineering
Cornel University
Ithaca, New York 1

Professor Fred C. Andrews
Department of Mathematics
University of Oregon
Eugene, Oregon 1

Professor Z.W. Birnbaum
Department of Mathematics
University of Washington
Seattle 5, Washington 1

Dr. David Blackwell
Department of Mathematical Sciences
University of California
Berkeley 4, California 1

Dr. Charles Boll
761 Partridge Avenue
Menlo Park, California 1

Professor William G. Cochran
Department of Statistics
Harvard University
2 Divinity Avenue, Room 311
Cambridge 38, Massachusetts 1

Dr. Joseph Daly
Bureau of the Census
Washington 25, D.C. 1

Miss Besse B. Day
Bureau of Ships, Code 310-B
Department of the Navy
Washington 25, D.C. 1

Dr. Walter L. Deemer, Jr.
Operations Analysis Div., DCE/O
Hq., U.S. Air Force
Washington 25, D.C. 1

Mr. James J. Fleming, Head
Operational Research Branch
U.S. Naval Research Laboratory
Washington 25, D.C. 1

Dr. Landis S. Gephart
Scientific Advisor
U.S. Army Research & Development
Liaison Group, APO No. 757
New York, New York 1

Mrs. Dorothy M. Gilford, Head
Logistics and Mathematical
Statistics Branch
Code 436
Office of Naval Research
Washington 25, D. C. 3

Mr. Harold Gumbel
Head, Operations Research Group
Code 01-2
Pacific Missile Range
Box 1
Point Mugu, California 1

Professor Donald Guthrie, Jr.
Stanford Research Institute
333 Ravenswood Ave.
Menlo Park, California 1

Dr. Ivan Hershner
Office, Chief of Research & Dev.
U.S. Army, Research Division 3E382
Washington 25, D.C. 1

Professor Harold Hotelling
Department of Statistics
University of North Carolina
Chapel Hill, N.C. 1

Professor Stanley Isaacson
4715 Pleasant Street
Des Moines, Iowa 1

Professor Carl K. Kossack
Department of Mathematics
Purdue University
Lafayette, Indiana 1

Professor W. H. Kruskal
Department of Statistics
The University of Chicago
Chicago 37, Illinois 1

Professor Eugene Lukacs
Department of Mathematics
Catholic University
Washington 16, D.C. 1

Dr. Craig Magwire
2954 Winchester Way
Rancho Cordova
California 1

Dr. Clifford Maloney
Chief, Statistics Branch
Chemical Corps Biological Warfare Lab.
Fort Detrick, Maryland 1

Professor Paul Meyer
Department of Mathematics
Washington State College
Pullman, Washington 1

Dr. Knox T. Millsaps
Chief Scientist
Air Force Missile Dev. Center
Holloman Air Force Base, New Mexico 1

D. E. Newnham
Chief, Industrial Engineering Division
Comptroller
Hdqrs., San Bernardino Air Materiel Area,
USAF, Norton Air Force Base, Calif. 1

Dr. Monroe Norden
College of Engineering
New York University
New York 53, N.Y. 1

Mr. R. H. Noyes
Office of Technical Plans
USASRDL
Ft. Monmouth, N.J. 1

Professor Edwin G. Olds
Department of Mathematics
College of Engineering & Sciences
Carnegie Institute of Technology
Pittsburgh 13, Pennsylvania 1

Dr. William R. Pabst
Quality Control Division
Bureau of Ordnance
Department of the Navy
Washington 25, D.C. 1

H. Walter Price, Chief
Reliability Branch, 750
Diamond Ordnance Fuze Laboratory
Room 105, Building 83
Washington 25, D.C. 1

Professor Ronald Pyke
Department of Mathematical Stat.
Columbia University
New York 27, New York 1

Professor Stanley Reiter
Department of Mathematics
Purdue University
Lafayette, Indiana 1

Dr. Paul Rider
Wright Air Development Center, WCRRM
Wright-Patterson A.F.B., Ohio 1

Professor Herbert Robbins
Dept. of Mathematical Statistics
Columbia University
New York 27, N.Y. 1

Professor Herman Rubin
Department of Mathematics
University of Oregon
Eugene, Oregon 1

Major Oliver A. Shaw
Air Force Office of Scientific Research
Attn: SRM, Room 2718
Temporary "X" Building
Washington 25, D.C. 2

Professor Laurie Snell
Department of Mathematics
Dartmouth College
Hanover, N.H. 1

Dr. Milton Sobel
Bell Telephone Laboratories
555 Union Blvd.
Allentown, Pennsylvania 1

Mr. G.P. Steck
Division 5511
Sandia Corp., Sandia Base
Albuquerque, New Mexico 1

Dr. Horace M. Trent
Head, Applied Mathematics Branch
Naval Research Laboratory
Washington 25, D.C. 1

Professor Donald Truax
Department of Mathematics
University of Oregon
Eugene, Oregon 1

Professor John W. Tukey
Department of Mathematics
Princeton University
Princeton, New Jersey 1

Dr. Harry Weingarten
Special Projects Office, SP2016
Navy Department
Washington 25, D. C. 1

Mr. Joseph Weinstein
Institute for Exploratory Research
USASRDL
Ft. Monmouth, N.J. 1

Dr. Irving Weiss
The Mitre Corp.
Bedford, Massachusetts 1

Dr. F. J. Weyl, Director
Mathematical Sciences Division
Office of Naval Research
Washington 25, D.C. 1

Dr. John Wilkes
Office of Naval Research
Code 200
Washington 25, D.C. 1

Mr. Silas Williams
Standards Branch, Proc. Div.
Office, DC/S for Logistics
Department of the Army
Washington 25, D.C. 1

Professor Jacob Wolfowitz
Department of Mathematics
Cornell University
Ithaca, New York 1

Mr. William W. Wolman
National Aeronautics and Space Adm'n.
1520 H Street, N.W., Code AAR
Washington 25, D.C. 1

Additional copies for project
leader and assistants and reserve
for future requirements 50