APPLIED MATHEMATICS AND STATISTICS LABORATORIES

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1. Introductory Discussion and Summary.

Let $\underset{\sim}{x} = (x_1, x_2, \ldots, x_n)$ be a normal random vector with zero expectation vector and with a variance-covariance matrix which has 1 for its diagonal elements and ρ for its off-diagonal elements. Consider the quantity

$$(1.1) \quad I_n(h;\rho) = (2\pi)^{-\frac{1}{2}n}\{1 + (n-1)\rho\}^{-\frac{1}{2}}(1-\rho)^{-\frac{1}{2}(n-1)}\int_h^\infty \cdots \int_h^\infty e^{-\frac{1}{2}Q(\underset{\sim}{x})} dx_1 \ldots dx_n,$$

where

$$(1.2) \quad \begin{aligned} Q(\underset{\sim}{x}) &= [\{1 + (n-1)\rho\}(1-\rho)]^{-1}[\{1 + (n-2)\rho\}\sum x_i^2 - 2\rho\sum_{j>i} x_i x_j] \\ &= (1-\rho)^{-1}[\sum x_i^2 - \rho\{1 + (n-1)\rho\}^{-1}(\sum x_i)^2] . \end{aligned}$$

Thus $I_n(h;\rho)$ is the probability that each of n normally distributed, equally correlated and standardized random variables with common correlation ρ shall not fall short of h. Clearly $1 - I_n(h;\rho)$ is also the distribution function of the random variable $\max_i x_i$, and this supplies one application (cf. [3]) of $I_n(h;\rho)$. A second application relates to the familiar one-factor model in factor analysis for the special case of equal weights [8]. Another situation in which knowledge of the distribution of $I_n(h;\rho)$ is important is in some models of test design in psychology. Other applications will arise or probably exist at present.

In a previous paper [8] (see also [8] for further references), $I_n(h;\rho)$ was expressed as the product of the density function of $\underset{\sim}{x}$ at the cut-off point $\underset{\sim}{h} = (h, h, \ldots, h)$ and an infinite power series in h. In this paper it will be shown for $h > 0$ that $I_n(h;\rho)$ can be expressed asymptotically as the product of the density function at h and an infinite series in negative powers of h. This result can be regarded as the generalization for $n > 1$ of the well-known asymptotic expansion of Mill's ratio

$$(1.3) \quad \int_x^\infty e^{-\frac{1}{2}t^2} dt \Big/ e^{-\frac{1}{2}x^2} \sim x^{-1}(1 - x^{-2} + 1\cdot 3x^{-4} - 1\cdot 3\cdot 5x^{-6} + \ldots)(x>0).$$

2. The Asymptotic Development of $I_n(h;\rho)$.

Under the transformation

$$(2.1) \quad \begin{aligned} y_1 &= [1 + (n-1)\rho]^{-\frac{1}{2}} \sum_{j=1}^{n} b_{1j}\, x_j , \\ y_i &= (1-\rho)^{-\frac{1}{2}} \sum_{j=1}^{n} b_{ij}\, x_j \qquad (i = 2, 3, \ldots, n) , \end{aligned}$$

where $((b_{ij}))$, $i, j = 1, 2, \ldots, n$, is orthogonal with $b_{1j} = n^{-\frac{1}{2}}$ $(j = 1, 2, \ldots, n)$, (1.1) reduces to

$$(2.2) \quad I_n(h;\rho) = (2\pi)^{-\frac{1}{2}n} \int \underset{R}{\cdots} \int e^{-\frac{1}{2}\sum y_i^2} dy_1 \ldots dy_n$$

with R defined by

$$(2.3)\quad R:\ [1 + (n-1)\,\rho]^{\frac{1}{2}}\,[n(1-\rho)]^{-\frac{1}{2}}\,y_1 + \sum_{j=2}^{n} b_{ji}\,y_j \geq (1-\rho)^{-\frac{1}{2}}\,h$$

$$(i = 1, 2, \ldots, n)$$

[8]. R is a polyhedral half-cone in $\underset{\sim}{y}$ - space with vertex at the point $(r_o, 0, 0, \ldots, 0)$, where

$$(2.4)\qquad r_o = [n/\{1 + (n-1)\,\rho\}]^{\frac{1}{2}}\,h\ ,$$

such that the angle between any two faces of the cone is arc cos - ρ; further the axis of the cone passes through the origin in $\underset{\sim}{y}$ - space. $I_n(h;\rho)$ is, then, the probability measure under an n - dimensional spherical normal distribution with unit standard deviation in any direction of a regular, symmetrically oriented polyhedral half-cone with common dihedral angle arc cos - ρ, and with vertex at a distance r_o from the center of the distribution. Let P be any point within the cone distant r from the center of the distribution, ξ from the axis of the cone and x from the vertex of the cone in a direction parallel to the axis. The probability-mass of an infinitesimal element of volume $d\tau$ at P is

$$(2.5)\quad (2\pi)^{-\frac{n}{2}}\,e^{-\frac{1}{2}r^2}\,d\tau = (2\pi)^{-\frac{1}{2}}\,e^{-\frac{1}{2}(r_o+x)^2}\,dx.(2\pi)^{-\frac{1}{2}(n-1)}\,e^{-\frac{1}{2}\xi^2}\,dS\ ,$$

where dS is the measure of an infinitesimal element in the $(n-1)$ - flat orthogonal to the axis of the cone and distant x from the vertex (cf.[5]). Consider the probability-mass in that portion of the cone (an infinitesimal "slab") demarcated by two adjoining $(n-1)$ - flats orthogonal to the axis of the cone and distant x and $x + dx$ from the vertex of the cone. It is easily shown that the intersection of the first of these two flats with the cone is a regular $(n-1)$ - dimensional simplex with centroid at the foot of the perpendicular from P to the axis of the cone and with edges of length

$$[2n\{1 + (n-1)\rho\}/(1-\rho)]^{\frac{1}{2}} x .$$

Let $K_N(\ell)$ denote the probability measure under an N-dimensional spherical normal distribution with unit standard deviation in any direction of a regular N-dimensional simplex with centroid at the center of the distribution and with edges of length ℓ. Then according to (2.5) the probability measure of the infinitesimal slab is

$$(2.6) \qquad (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}(r_o+x)^2} dx \cdot K_{n-1}\left[\left(\frac{2n\{1 + (n-1)\rho\}}{1-\rho}\right)^{\frac{1}{2}} x\right] .$$

consequently, the probability measure of the cone is

$$(2.7) \qquad \begin{aligned} I_n(h;\rho) &= \int_o^\infty (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}(r_o+x)^2} K_{n-1}\left[\left(\frac{2n\{1 + (n-1)\rho\}}{1-\rho}\right)^{\frac{1}{2}} x\right] dx \\ &= (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}r_o^2} \int_o^\infty e^{-r_o x} e^{-\frac{1}{2}x^2} K_{n-1}(\lambda x)\, dx , \end{aligned}$$

where

$$\lambda \equiv \lambda_n(\rho)$$

$$(2.8) \qquad = [2n\{1 + (n-1)\rho\}/(1-\rho)]^{\frac{1}{2}}$$

and r_o is given by (2.4). Formula (2.7) which is of considerable intrinsic interest may be used also to develop the required asymptotic expansion of $I_n(h;\rho)$ for $h > 0$.[1/]

The K-functions are closely related to Godwin's G-function [1], [2] introduced in connection with the distribution of the absolute mean deviation in normal samples, and some further statistical applications of the functions have been discussed in [4] and [5]. Clearly, $K_N(x)$ is bounded by 1. Again, it has been shown elsewhere [7] that $K_N(x)$ has a power series expansion with infinite radius of convergence. Consequently, Watson's lemma [9] (p. 236) may be used to obtain a valid asymptotic expansion for the integral in (2.7) by expanding $\exp(-x^2/2)\, K_{n-1}(\lambda x)$ in its Taylor series at $x = 0$ and integrating term by term. In fact, let

$$(2.9) \qquad \psi_n(x) \equiv \psi_n(x;\lambda) \equiv e^{-\frac{1}{2}x^2} K_{n-1}(\lambda x) = \sum_{i=0}^{\infty} c_{n-1,i}\, x^i/i! \,,$$

[1/] The center of the distribution is interior or exterior to the half-cone according as to whether it is within or without the half-cone, corresponding to the cases $h \geq 0$ and $h < 0$. The integral formula for $I_n(h;\rho)$ in (2.7) is valid for all h, but for the asymptotic expansion developed subsequently (equ. (2.22)) $h > 0$. (The case $h < 0$ is not likely to be of practical interest, while $I_n(0;\rho)$ is known to be equal to the normed measure of a regular $(n-1)$ - dimensional spherical simplex with common dihedral angle arc cos $-\rho$. The reader is referred to [9] where tables of each normed measures are provided for $n = 1(1)\ 51 - i$ and $\rho = 1/i$, $i = 1(1)\ 12$.)

where the $c_{n-1,i}$ are functions of λ (and therefore of ρ). Then (2.7) gives with the aid of Watson's lemma,

$$I_n(h;\rho) \sim (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}r_o^2} \sum_{i=o}^{\infty} c_{n-1,i}\, r_o^{-(i+1)} . \tag{2.10}$$

This is the required formula. It should be noted that the probability density in the original distribution at the point $(h, h, \ldots, h)$ is

$$(2\pi)^{-\frac{1}{2}n} \{1 + (n-1)\rho\}^{-\frac{1}{2}} (1-\rho)^{-\frac{1}{2}(n-1)} e^{-\frac{1}{2}r_o^2} , \tag{2.11}$$

thereby justifying the assertion at the end of the introductory Section.

It now remains to determine the coefficients $c_{n-1,i}$ in (2.10). On differentiating (2.9) j times at $x = 0$ we obtain after some simplification

$$\begin{aligned} c_{n-1+2k} &= \psi_{n-1}^{(n-1+2k)}(0) \\ &= \sum_{s=o}^{k} \left(-\tfrac{1}{2}\right)^{k-s} \frac{(n-1+2k)!}{(k-s)!} \lambda^{n-1+2s} a_{n-1,n-1+2s} , \end{aligned} \tag{2.12}$$

$$c_{n-1,m} = 0 \qquad (m = 0, 1, 2, \ldots, n-2) ,$$

where $\psi_{n-1}^{(n-1+2k)}(0)$ is the $(n-1+2k)$th derivative of $\psi_{n-1}(x)$ at $x = 0$ and the a's are defined by

$$K_N(x) = \sum_{j=o}^{\infty} a_{N,j}\, x^j \qquad (N = 0, 1, 2, \ldots)$$

($a_{N,j} = K_N^{(j)}(0)/j!$). In the derivation of (2.12) use has been made of the fact that

$$(2.13)\qquad \begin{aligned} a_{N,j} &= 0 \qquad && (j = 0, 1, 2, \ldots , N - 1) , \\ a_{N,N+2r+1} &= 0 \qquad && (r = 0, 1, 2, \ldots) . \end{aligned}$$

Formula (2.13) in its turn derives by induction from the following recursion relationship between the a's proved elsewhere [7]:

$$(2.14)\quad a_{N,s} = (2s)^{-1} \{(N + 1)/(N\pi)\}^{\frac{1}{2}} \sum_{q=o}^{[(s-1)/2]} \{-4N(N + 1)\}^{-q} a_{N-1,s-1-2q}/q! \qquad (s = 1, 2, \ldots) ,$$

$[(s-1)/2]$ denoting, as usual, the integral part of $(s-1)/2$. Though (2.14) may be exploited to derive explicit expressions for the non-negative a's these are more easily obtained recursively by repeated application of (2.14) on noting that

$$(2.15)\qquad \begin{aligned} a_{o,j} &= 0 \qquad && (j = 1, 2, \ldots) , \\ &= 1 \qquad && (j = 0) . \end{aligned}$$

This yields for the first three non-negative $a_{n-1,j}$,

$$(2.16)\qquad a_{n-1,n-1} = \frac{n^{\frac{1}{2}}}{2^{n-1}\,\pi^{\frac{1}{2}(n-1)}} \; \frac{1}{(n - 1)!} ,$$

$$(2.17) \qquad a_{n-1,n+1} = - \frac{n^{\frac{1}{2}}}{2^{n-1} \pi^{\frac{1}{2}(n-1)}} \quad \frac{1}{4(n+1)!} ,$$

$$(2.18) \qquad a_{n-1,n+3} = \frac{n^{\frac{1}{2}}}{2^{n-1} \pi^{\frac{1}{2}(n-1)}} \quad \frac{(n-1)(n^2+7n-6)}{32n} \quad \frac{1}{(n+3)!}$$

((2.14) shows that the non-negative a's oscillate in sign).

On applying (2.16), (2.17) and (2.18) in (2.12), the first three non-negative c's are obtained:

$$(2.19) \qquad \begin{aligned} c_{n-1,n-1} &= (n-1)!\, \lambda^{n-1} a_{n-1,n-1} \\ &= n^{\frac{1}{2}}\, 2^{-(n-1)}\, \pi^{-\frac{1}{2}(n-1)}\, \lambda^{n-1} , \end{aligned}$$

$$(2.20) \qquad \begin{aligned} c_{n-1,n+1} &= (n+1)!\, \{-\tfrac{1}{2} \lambda^{n-1} a_{n-1,n-1} + \lambda^{n+1} a_{n-1,n+1}\} \\ &= -n^{\frac{1}{2}}\, 2^{-(n-1)}\, \pi^{-\frac{1}{2}(n-1)} \{\tfrac{1}{2} n(n+1) \lambda^{n-1} + \tfrac{1}{4} \lambda^{n+1}\} , \end{aligned}$$

$$c_{n-1,n+3} = (n+3)!\,\{\tfrac{1}{8}\lambda^{n-1}a_{n-1,n-1} - \tfrac{1}{2}\lambda^{n+1}a_{n-1,n+1} + \lambda^{n+3}a_{n-1,n+3}\}$$

$$(2.21)\qquad = n^{\frac{1}{2}}\,2^{-(n-1)}\,\pi^{-\frac{1}{2}(n-1)}\{\tfrac{1}{8}n(n+1)(n+2)(n+3)\lambda^{n-1} + \tfrac{1}{8}(n+2)(n+3)\lambda^{n+1} + \tfrac{1}{32}\frac{(n-1)(n^2+7n-6)}{n}\lambda^{n+3}\}\ .$$

Thus from (2.10),

$$(2.22)\qquad I_n(h;\rho) \sim (2\pi)^{-\frac{1}{2}}\,e^{-\frac{1}{2}r_o^2}\{c_{n-1,n-1}\,r_o^{-(n+1)} + c_{n-1,n+1}\,r_o^{-(n+3)} + c_{n-1,n+3}\,r_o^{-(n+5)} + \ldots\},$$

where the first three coefficients in the asymptotic expansion are given by (2.19), (2.20) and (2.21)(further coefficients may be obtained in the manner shown). A slightly more convenient form of (2.22) is

$$I_n(h;\rho) \sim (\tfrac{1}{2}\,n)^{\frac{1}{2}}\,\pi^{-\frac{1}{2}n}\,e^{-\frac{1}{2}r_o^2}\,(t/r_o)^{n-1}\,r_o^{-1}$$

$$(2.23)\qquad \times\Big[1 - \{\tfrac{1}{2}(n)_2 + t^2\}r_o^{-2} + \{\tfrac{1}{8}(n)_4 + \tfrac{1}{2}(n+2)_2\,t^2 + \tfrac{1}{2}(n-1)(n^2+7n-6)n^{-1}\,t^4\}r_o^{-4} - \ldots\Big],$$

where

$$(2.24)\qquad t \equiv t_n(\rho) = \lambda/2 = [n\{1 + (n-1)\rho\}/2(1-\rho)]^{\frac{1}{2}}$$

and $(n)_m$ denotes $n(n+1) \cdots (n+m-1)$. It will be noted that the present asymptotic expansion is particularly suitable for large r_o (i.e., the cut-off point is not near the center of the distribution) and algebraically small ρ.

Finally, observe that for $n = 1$ (2.22) reduces to (1.3), since $\psi_o(x) = \exp(-x^2/2)$ and

$$c_{o,2j} = (-\tfrac{1}{2})^j (2j)!/j! \quad (2.25)$$

(The polyhedral half-cone is here the interval (h,∞).) For $n = 2$, (2.22) reduces to

$$I_2(h;\rho) \sim \pi^{-1} e^{-\frac{1}{2}r_o^2} t[1-(3+t^2)r_o^{-2} + (15 + 10t^2 + 3t^4)r_o^{-4} - \cdots]. \quad (2.26)$$

This agrees with a formula obtained previously [6] for the probability measure, $W(h;\alpha)$, under a standardized circular normal distribution of a sector of angle α, vertex at a distance h from the center of the distribution and with one arm of the sector passing through the latter point. The relationship between I_2 and W is

$$I_2(h;\rho) = 2W(h;\theta/2) \quad (2.27)$$

where $\theta = 2 \text{ arc tan } t = 2 \text{ arc tan}\{(1+\rho)/(1-\rho)\}^{\frac{1}{2}}$. It has been shown in [6] that the bivariate normal integral for arbitrary cut-off point may be expressed in terms of the difference of two W-functions (and therefore of two I_2-functions).

3. The Accuracy of the Asymptotic Expansion.

In this section we obtain an upper bound to the error induced by taking the first m terms of the asymptotic expansion as an approximation to $I_n(h;\rho)$. In particular, this allows a weaker upper bound to be obtained, to the effect that the above error is numerically not greater than the (m+1)th term of the expansion for all h.

Let ϕ be the angle between the axis of the half-cone and the line joining any point P and the vertex of the cone. Then (using the notation of Section 2)

$$r^2 = r_o^2 + \xi^2 + 2r_o\,\xi\,\cos\phi\,,$$

and the probability-mass of an infinitesimal volume-element of content $d\tau$ as P is

$$(2\pi)^{-\frac{1}{2}n}\exp[-\tfrac{1}{2}r^2]\,d\tau = (2\pi)^{-\frac{1}{2}n}\,.$$

(3.1)

$$.\exp[-\tfrac{1}{2}(r_o^2 + \xi^2 + 2r_o\xi\,\cos\phi)]\xi^{n-1}\,d\xi d\omega\,,$$

where $d\omega$ is the solid angle subtended at the center of the distribution by the volume-element (or, equivalently, the surface-content of an infinitesimal element on the surface of a unit sphere whose center coincides with the center of the distribution). Thus the probability-mass of the half-cone is

$$(3.2)\quad I_n(h;\rho) = (2\pi)^{-\frac{1}{2}n}\,e^{-\frac{1}{2}r_o^2}\int_o^\infty\int_\Omega e^{-(r_o\cos\phi)\xi}\,\xi^{n-1}\,e^{-\frac{1}{2}\xi^2}\,d\xi d\omega\,,$$

where Ω is the $(n-1)$-dimensional regular spherical simplex (with common dihedral angle $\arccos - \rho$) formed by the intersection of the half-cone and the surface of the unit sphere. Again, if

$$G_{n-1}(\xi) = \xi^{n-1} e^{-\frac{1}{2}\xi^2} ,$$

then the derivatives of $G_{n-1}(\xi)$ at the origin, $G_{n-1}^{(q)}(0)$, are given by

$$G_{n-1}^{(n-1+2i)}(0) = (-1)^i \frac{(n - 1 + 2i)!}{2^i i!} \qquad (i = 0, 1, 2, \dots)$$

with all other derivatives vanishing. Therefore, repeated integration by parts yields

$$(3.3) \quad \int_0^\infty e^{-(r_0 \cos \phi)\xi} G_{n-1}(\xi)d\xi = \sum_{i=0}^{m-1} (-1)^i \frac{(n-1+2i)}{2^i i!} \frac{1}{(r_0 \cos \phi)^{n+2i}} + R_m(r_0 \cos \phi) ,$$

where

$$(3.4) \quad \begin{aligned} R_m(r_0 \cos \phi) &= (r_0 \cos \phi)^{-(n+2m-2)} \int_0^\infty e^{-(r_0 \cos \phi)\xi} G_{n-1}^{(n+2m-2)}(\xi)d\xi \\ &= (r_0 \cos \phi)^{-(n+2m-1)} \int_0^\infty e^{-(r_0 \cos \phi)\xi} G_{n-1}^{(n+2m-1)}(\xi)d\xi \end{aligned}$$

after a further single integration by parts. On using (3.3) and (3.4) in (3.2),

$$I_n(h;\rho) = (2\pi)^{-\frac{1}{2}n} e^{-\frac{1}{2}r_o^2} \left\{ \sum_{i=o}^{m-1} (-1)^i \frac{(n-1+2i)!}{2^i i!} \alpha_{n,i} r_o^{-(n+2i)} \right. \tag{3.5}$$

$$\left. + \int_\Omega R_m(r_o \cos \phi) d\omega \right\} ,$$

where

$$\alpha_{n,i} = \int_\Omega \sec^{n+2i} \phi \; d\omega \; . \tag{3.6}$$

In (3.5), the error after m terms is

$$E_m = (2\pi)^{-\frac{1}{2}n} e^{-\frac{1}{2}r_o^2} \int_\Omega R_m(r_o \cos \phi) d\omega \; . \tag{3.7}$$

An upper bound to $|E_m|$ can be obtained from an upper bound to $R_m(r_o \cos \phi)$ in (3.4). The latter upper bound is itself obtained by deriving first an upper bound to $G_{n-1}^{(n+2m-1)}(\xi)$ for $\xi \geq 0$. If, then,

$$|G_{n-1}^{(n+2m-1)}(\xi)| \leq A_{n-1,2m} \; , \tag{3.8}$$

(3.4) gives for $r_o > 0$

$$|R_m(r_o \cos \phi)| < A_{n-1,2m} (r_o \cos \phi)^{-(n+2m)} \; , \tag{3.9}$$

whence by (3.7)

(3.10)
$$|E_m| \leq (2\pi)^{-\frac{1}{2}n} e^{-\frac{1}{2}r_o^2} A_{n-1,2m} \int_\Omega (r_o \cos \phi)^{-(n+2m)} d\omega$$
$$= A_{n-1,2m} (2\pi)^{-\frac{1}{2}n} e^{-\frac{1}{2}r_o^2} \alpha_{n,m} r_o^{-(n+2m)} ,$$

which is proportional to the $(m + 1)$th term of the series

(3.11)
$$\sum_{i=o}^{\infty} (-1)^i \frac{(n - 1 + 2i)!}{2^i i!} \alpha_{n,i} r_o^{-(n+2i)} .$$

Consequently, (3.11) is a valid asymptotic expansion when $r_o > 0$ of $I_n(h;\rho)$. Moreover, the series (3.11) must be identical with the series (2.22), since a given function determines uniquely (if at all) a series of the form $\sum c_p/r_o^p$, so that (3.10) provides an upper bound to the error in using (2.22).

We now proceed to determine a value[2/] for $A_{n-1,2m}$. Let

(3.12)
$$\xi^{n-1} = \beta_{n-1,o} H_o(\xi) + \beta_{n-1,1} H_1(\xi) + \cdots + \beta_{n-1,n-1} H_{n-1}(\xi) ,$$

where $H_j(\xi)$ are the Tchebycheff-Hemite polynomials orthogonal to the weight function $\exp(- \xi^2/2)$ and normalized so that the coefficient of ξ^j in $H_j(\xi)$ is 1. On multiplying (3.12) by $H_j(\xi) \exp(- \xi^2/2)$, and integrating over the real line, we find

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2/ That $A_{n-1,2m} < \infty$ is evident from the fact that all derivatives of $G_{n-1}(\xi)$ are products of polynomials in ξ and $\exp(-\xi^2/2)$.

(3.13) $$\beta_{n-1,j} = \int_{-\infty}^{\infty} \xi^{n-1} H_j(\xi)\, e^{-\frac{1}{2}\xi^2} \Big/ \int_{-\infty}^{\infty} H_j^2(\xi)\, e^{-\frac{1}{2}\xi^2}\, d\xi \ .$$

The value of the denominator in (3.13) is well-known to be $\sqrt{2\pi}\ j!$. In order to evaluate the numerator, define

$$\gamma_{n-1,j} = \int_{-\infty}^{\infty} \xi^{n-1} H_j(\xi)\, e^{-\frac{1}{2}\xi^2}\, d\xi \ .$$

Integration by parts gives the recursion relationship

(3.14) $$\gamma_{n-1,j} = (n-1)\,\gamma_{n-2,j-1} \ ,$$

and on successive application of (3.14)

$$\gamma_{n-1,j} = (n-1)(n-2)\cdots(n-j)\,\gamma_{n-1-j,o}$$

$$= (n-1)(n-2)\cdots(n-j)\int_{-\infty}^{\infty} \xi^{n-1-j}\, e^{-\frac{1}{2}\xi^2}\, d\xi \ ,$$

whence

(3.15) $$\gamma_{n-1,j} = (n-1)(n-2)\cdots(n-j)\, 2^{\frac{1}{2}(n-j)}\, \Gamma\left(\tfrac{1}{2}(n-j)\right) \qquad (n-1-j \text{ even}) \ ,$$

$$= 0 \qquad (n-1-j \ \text{ odd}) \ .$$

On substituting (3.15) in (3.13),

$$\beta_{n-1,j} = \frac{(n-1)!}{2^{\frac{1}{2}(n-1-j)}[\frac{1}{2}(n-1-j)]!\,j!} \qquad (n-1-j \text{ even}) ,$$

(3.16

$$= 0 \qquad (n-1-j \text{ odd}) .$$

Reverting to (3.12),

$$G_{n-1}(\xi) = \xi^{n-1} e^{-\frac{1}{2}\xi^2}$$

$$= \sum_{j=o}^{n-1} \beta_{n-1,j} H_j(\xi) e^{-\frac{1}{2}\xi^2} ,$$

and therefore

$$(3.17) \qquad G_{n-1}^{(n-1+2m)}(\xi) = \sum_{j=o}^{n-1} \beta_{n-1,j} H_{j+2m}(\xi) e^{-\frac{1}{2}\xi^2}$$

on recalling that

$$(3.18) \qquad \frac{d^p}{d\xi^p} e^{-\frac{1}{2}\xi^2} = (-1)^p H_p(\xi) e^{-\frac{1}{2}\xi^2} .$$

An upper bound to $|H_{j+2m}(\xi)| e^{-\frac{1}{2}\xi^2}$ in (3.17) is readily deduced from the identity

$$e^{-\frac{1}{2}\xi^2} = \int_{-\infty}^{\infty} e^{i\xi x} \cdot (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}x^2} dx$$

for real ξ. Hence on applying (3.18),

$$(-1)^p H_p(\xi) e^{-\frac{1}{2}\xi^2} = \int_{-\infty}^{\infty} (ix)^p e^{i\xi x} (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}x^2} dx ,$$

from which we obtain

$$|H_p(\xi)| e^{-\frac{1}{2}\xi^2} \leqq (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} |x|^p e^{-\frac{1}{2}x^2} dx$$

(3.19)

$$= \pi^{-\frac{1}{2}} 2^{\frac{1}{2}p} \Gamma[\tfrac{1}{2}(p+1)] .$$

Thus from (3.16), (3.17) and (3.19),

(3.20)

$$|G_{n-1}^{(n-1+2m)}(\xi)| \leqq \pi^{-\frac{1}{2}} \sum_j{}' \frac{(n-1)!}{2^{\frac{1}{2}(n-1-j)} [\frac{1}{2}(n-1-j)]! j!} 2^{m+\frac{1}{2}j} \Gamma[m + \tfrac{1}{2}(j+1)],$$

$\sum_j{}'$ denoting summation over all non-negative integral $j \leqq n-1$ such that $n-1-j$ is even.

If n is odd, set $j = 2i$ in (3.20). Then

$$|G_{n-1}^{(n-1+2m)}(\xi)| \leqq \pi^{-\frac{1}{2}} \sum_{i=o}^{(n-1)/2} \frac{(n-1)!}{2^{\frac{1}{2}(n-1-2i)} [\frac{1}{2}(n-1)-i]! (2i)!} .$$

$$\cdot 2^{m+i} \Gamma(m + i + \tfrac{1}{2}) ,$$

and, on using the duplication formula for the gamma function in the form

$$\pi^{-\frac{1}{2}} \Gamma\left(m + i + \tfrac{1}{2}\right) = (2m + 2i)! \Big/ \{(m + i)!\ 2^{2m+2i}\} ,$$

the latter inequality simplifies to

$$|G_{n-1}^{(n-1+2m)}(\xi)| \leqq \frac{(n-1)!}{2^{m+\frac{1}{2}(n-1)}} \sum_{i=0}^{(n-1)/2} \frac{(2m + 2i)!}{(m + i)!\ (2i)!} \frac{1}{(\frac{n-1}{2} - i)!} \tag{3.21}$$

$(n = 1, 3, \ldots)$.

Similarly, if n is even, set $j = 2i + 1$ in (3.20). Then

$$|G_{n-1}^{(n-1+2m)}(\xi)| \leqq \pi^{-\frac{1}{2}} \sum_{i=0}^{(n-2)/2} \frac{(n-1)!}{2^{\frac{1}{2}(n-2-2i)} [\frac{1}{2}(n-2)-i]!\ (2i+1)!} \, . \tag{3.22}$$

$$\cdot 2^{m+i+\frac{1}{2}} \Gamma(m + i + 1) ,$$

and, on using gamma duplication formula in the form

$$\pi^{-\frac{1}{2}} \Gamma(m + i + 1) = (2m + 2i + 1)! \Big/ \{ \Gamma\left(m + i + \tfrac{3}{2}\right) 2^{2m+2i+1}\} ,$$

the last inequality reduces to

$$|G_{n-1}^{(n-1+2m)}(\xi)| \leq \frac{(n-1)!}{2^{m+\frac{1}{2}(n-1)}} \sum_{i=o}^{(n-2)/2} \frac{(2m+2i+1)!}{\Gamma(m+i+\frac{3}{2})(2i+1)!} \tag{3.23}$$

$$\cdot \frac{1}{(\frac{n-2}{2}-i)!} \qquad (n = 2, 4, \dots) .$$

Formulae (3.21) and (3.23) provide the required inequalities in the sense that their right-hand members (refer to (3.8)) may be substituted for $A_{n-1,2m}$ in (3.10) to supply the desired upper bound for the error after m terms. A weaker upper bound may be obtained by noting that in (3.21)

$$(n-1)!\ (2m+2i)!\Big/(2i)! = (n-1)!\ (2i+1)(2i+2) \cdots (2m+2i)$$

$$\leq (n-1+2m)! ,$$

whence[3/]

$$|G_{n-1}^{(n-1+2m)}(\xi)| \leq \frac{(n-1+2m)!}{2^m\ m!} \cdot \frac{\frac{1}{2}(n+1)}{2^{\frac{1}{2}(n-1)}} \tag{3.24}$$

$$\leq \frac{(n-1+2m)!}{2^m\ m!} \qquad (n = 1, 3, \dots) ,$$

3/ There are $(n+1)/2$ terms in the series (3.21), and to obtain (3.24) the largest term of these $(n+1)/2$ terms is substituted for each term.

since $(n+1)/2^{\frac{1}{2}(n+1)} \leqq 1$ for all odd n. Similarly, for n even, observe that in (3.23)

$$(n-1)!\,(2m+2i+1)!/(2i+1)! = (n-1)!\,(2i+2)(2i+3)\cdots(2m+2i+1)$$

$$\leqq (n-1+2m)!\,,$$

whence

$$|G_{n-1}^{(n-1+2m)}(\xi)| \leqq \frac{(n-1+2m)!}{2^m \Gamma(m+3/2)} \frac{\frac{1}{2}n}{2^{\frac{1}{2}(n-1)}} < \frac{(n-1+2m)!}{2^m\, m!} \frac{\frac{1}{2}n}{2^{\frac{1}{2}(n-1)}}$$

(3.25)

$$< \frac{(n-1+2m)!}{2^m\, m!}$$

$$(n = 2, 4, \ldots),$$

since $n/2^{\frac{1}{2}(n+1)} < 1$ for all even n. An upper bound to $|G_{n-1}^{(n-1+2m)}(\xi)|$ is thus $(n-1+2m)!/2^m\, m!$ for all n. This upper bound may be substituted for $A_{n-1,2m}$ in (3.10), thereby proving that the numerical error after n terms is less than the absolute value of the $(m+1)$th term. It should be noted, however, that according to (3.25) this can be improved for even n by replacing $A_{n-1,2m}$ by $(n-1+2m)!/\{2^m \Gamma(m+3/2)\}$.

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