AN OBSTACLE-COURSE PROBLEM: I

BY

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Summary.

An obstacle course is presented in which n obstacles are given, together with p_i, the probability of successfully overcoming the i^{th} obstacle $(0 < p_i < 1)$, and $v_i > 0$, the value of overcoming it, for each obstacle $i=1,2,\ldots,n$. A runner is permitted to choose $r \leq n$ (r fixed) obstacles and to order them, so that he maximizes his expected value, on the assumption that once he fails to overcome an obstacle he receives only the points previously earned.

The optimal ordering of the chosen r is a simple function of the obstacle parameters. The optimal choice of which r to attempt is more complicated, and three alternative computational procedures are given, together with a proof that each will terminate at the optimal r, and an upper bound on the number of comparisons required for each.

1. Ordering.

Notation: Let $V(1;2;\ldots;r)$ = the expected value attained if obstacles $1,2,\ldots,r$ are attempted in that order.

Lemma 1.

$$V(1;2;\ldots;r) = p_1v_1 + p_1p_2v_2 + \cdots + p_1p_2\cdots p_rv_r$$

Proof. The outcome is the sum of the random variables, having value v_i with probability $p_1 \cdots p_i$ and value zero otherwise. The expectation of this sum is the sum of the expectations, and is given above.

In investigating optimal ordering for a fixed r obstacles, we would like to know first what the effect is of exchanging obstacles i and $i+1$. This result is given in Lemma 2.

Notation. Let $q_i = 1-p_i$ and $\varphi_i = \dfrac{p_i v_i}{q_i}$ for $i=1,\ldots,n$.

Lemma 2. $V(1;2;\ldots;i-1;i;i+1;i+2;\ldots;r)-V(1;2;\ldots;i-1;i+1;i;i+2;\ldots;r) =$

a) $p_1 \cdots p_{i-1}\, q_i\, q_{i+1}[\varphi_i-\varphi_{i+1}] =$

b) $(\pi_{j<i} p_j)\, q_i q_{i+1}[\varphi_i-\varphi_{i+1}]$.

Proof.

$$V(1;2;\ldots;i-1;i;i+1;i+2;\ldots;r)-V(1;2;\ldots;i-1;i+1;i;i+2;\ldots;r) =$$

$$p_1v_1+p_1p_2v_2+\cdots+p_1p_2\cdots p_{i-1}v_{i-1}+p_1p_2\cdots p_{i-1}p_iv_i+p_1\cdots p_{i-1}p_ip_{i+1}v_{i+1}+\cdots+ p_1\cdots p_{i-1}p_i\cdots p_rv_r$$

$$-[p_1v_1+p_1p_2v_2+\cdots+p_1p_2\cdots p_{i-1}v_{i-1}+p_1p_2\cdots p_{i-1}p_{i+1}v_{i+1}+p_1\cdots p_{i-1}p_{i+1}p_iv_i+\cdots+ p_1\cdots p_{i-1}p_i\cdots p_rv_r]$$

$$= p_1\cdots p_{i-1}p_iv_i+p_1\cdots p_{i-1}p_ip_{i+1}v_{i+1}-p_1p_2\cdots p_{i-1}p_{i+1}v_{i+1}-p_1\cdots p_{i-1}p_{i+1}p_iv_i$$

$$= (p_1\cdots p_{i-1})[p_iv_i+p_ip_{i+1}v_{i+1}-p_{i+1}v_{i+1}-p_{i+1}p_iv_i]$$

$$= (p_1\cdots p_{i-1})[(p_i-p_{i+1}p_i)v_i-(p_{i+1}-p_ip_{i+1})v_{i+1}]$$

$$= (p_1\cdots p_{i-1})[(p_i)(1-p_{i+1})v_i-(p_{i+1})(1-p_i)(v_{i+1})] =$$

$$= (p_1\cdots p_{i-1})q_iq_{i+1}\left[\frac{p_iv_i}{q_i} - \frac{p_{i+1}v_{i+1}}{q_{i+1}}\right]$$

$$= (p_1\cdots p_{i-1})q_iq_{i+1}[\varphi_i-\varphi_{i+1}] ,$$

which is a). But this can be rewritten as

$$= (\pi_{j<i} p_j)q_iq_{i+1}[\varphi_i-\varphi_{i+1}],$$

which is b).

Theorem 1. Given any choice of obstacles $\{1,2,\ldots,r\}$, the order $(1,2,\ldots,r)$ is optimal if and only if $\varphi_i \geq \varphi_{i+1}$ $i=1,\ldots,r-1$.

Proof. If $(1,2,\ldots,r)$ is an optimal order, then for each i, $i=1,\ldots,r-1$, $0 \leq V(1;2;\ldots;i-1;i;i+1;\ldots;r)-V(1;2;\ldots;i-1;i+1;i;i+2;\ldots;r) = (\pi_{j<i}p_i)q_i q_{i+1}[\varphi_i-\varphi_{i+1}]$, so $\varphi_i \geq \varphi_{i+1}$ since all the other factors are positive. Conversely the condition $\varphi_i \geq \varphi_{i+1}$ defines the order uniquely up to indifference if $\varphi_i = \varphi_{i+1}$ for some i, and is therefore a sufficient condition for optimality.

2. Choice.

Lemma 3. $V(1;2;\ldots;s-1;s;s+1;\ldots;r)-V(s;1;2;\ldots;s-1;s+1;\ldots;r) = \sum_{i=1}^{s-1} (\pi_{j<i}p_j)q_i q_s[\varphi_i-\varphi_s]$.

Proof.

$$
\begin{aligned}
V(1;2;\ldots;s-1;s;s+1;\ldots;r)-V(s;1;2;\ldots;s-1;s+1;\ldots;r) = \\
[V(1;2;\ldots;s-1;s;s+1;\ldots;r) \\
-V(1;2;\ldots;s-2;s;s-1;s+1;\ldots;r)]+[V(1;2;\ldots;s-2;s;s-1;s+1;\ldots;r) \\
-V(1;2;\ldots;s-3;s;s-2;s-1;s+1;\ldots;r)] \\
+\cdots+[V(1;s;2;\ldots;s-1;s+1;\ldots r)-V(s;1;2;\ldots;s-1;s+1;\ldots;r)] = \\
\sum_{i=1}^{s-1} (\pi_{j<i}p_j)q_i q_s[\varphi_i-\varphi_s]
\end{aligned}
$$

by successive use of lemma 2b.

Notation. If A is a set of obstacles, let $V(A)$ be the expected value of the set, optimally ordered.

Theorem 2: [Fundamental Result]. Every optimal choice of $r+1$ obstacles includes an optimal choice of r obstacles.

Proof. Suppose the contrary were so. Then for some $r+1$ there is an optimal choice R^* which contains no optimal choice of r obstacles. Consider some set R which is an optimal choice of r obstacles. We will show that R^* can be improved (strictly) by inclusion of R, which contradicts the hypothesis that R^* is optimal.

Since R has r elements and R^* has $r+1$ elements, $R^* \cap R^c \neq \Omega$. Order R^* according to φ (which we know to be optimal ordering) and select one of those elements in $R^* \cap R^c$ with the highest φ. Suppose this obstacle is the s^{th} in R^*. We will show that $R \cup \{s\}$ has a higher expected value than R^*.

Case I: $s=1$. Then we have

$$V(R^*) = p_s v_s + p_s\ V(2;3;\ldots;r+1) < p_s v_s + p_s\ V(R) \leq V(R \cup \{s\})$$

since $\{2,3,\ldots,r+1\}$ is not an optimal r, by assumption.

Case II: $s > 1$. We will use the same idea for this proof, moving the s^{th} obstacle to the first positive in R^*, inserting R for $R^*-\{s\}$, and moving s back to its proper position according to φ order. If we can show that the loss L in moving s to the first position is not greater than the gain G in moving it back, the theorem will be proved.

By definition, $L = V(1;2;\ldots;s-1;s;s+1;\ldots;r+1)-V(s;1;2;\ldots;s-1;s+1;\ldots;r+1)$, which, by lemma 3, equals $\sum_{i=1}^{s-1} (\pi_{j<i}\ p_j)q_i q_s[\varphi_i-\varphi_s]$, so we have

$$(1) \quad L = q_s[q_1(\varphi_1-\varphi_s)+(1-q_1)q_2(\varphi_2-\varphi_s)+\cdots+(1-q_1)\cdots(1-q_{s-2})q_{s-1}(\varphi_{s-1}-\varphi_s)].$$

Each of obstacles $1,2,\ldots,s-1$ appear in both R and R^*, but there may be others in R. Suppose that between obstacles $i-1$ and i in R^* we have new obstacles $(i,1);(i,2);\ldots;(i,n_i)=i$, ordered, of course, according to φ, all (i,j)'s in R. Then by definition we have

$$G = V(1,1;1,2;\ldots;1,n_1;2,1;2,2;\ldots;s,1;s,2;\ldots;s,n_s=s;\ s^*;\ldots;r+1)$$
$$-V(s;1,1;1,2;\ldots;1,n_1;2,1;\ldots;s,1;s,2;\ldots;s,n_s-1;\ s^*;\ldots;r+1)\ .$$

Again we can use lemma 3 to express G, and we have

$$G = \sum_{(i,k)=(1,1)}^{(s,n_s-1)} (\pi_{(j,\ell)<(i,k)} p_{j,\ell}) q_s q_{i,k} [\varphi_{i,k}-\varphi_s]$$

$$\geq q_s \sum_{(i,k)=(1,1)}^{(s-1,n_{s-1})} (\pi_{(j,\ell)<(i,k)} p_{j,\ell}) q_{i,k} [\varphi_{i,k}-\varphi_s]\ .$$

(Here $(j,\ell)<(i,k)$ is according to lexicographic order: $j \leq i$, and, if $j=i$, $\ell < k$).

Now by assumption we have $\varphi_{i,k} \geq \varphi_i \geq \varphi_s$ for $k=1,\ldots,n_i$, so

$$G \geq q_s \sum_{i=1}^{s-1} [\varphi_i-\varphi_s] \sum_{k=1}^{n_i} (\pi_{(j,\ell)<(i,k)} p_{j,\ell}) q_{i,k}\ .$$

Now let $r_i = p_{i,1}p_{i,2} \cdots p_{i,n_i-1}$ so $0 < r_i \leq 1$ and

$$G \geq q_s \sum_{i=1}^{s-1} [\varphi_i-\varphi_s](p_1r_1\cdots p_{i-1}r_{i-1})((1-p_{i,1})+p_{i,1}(1-p_{i,2})+\cdots+ p_{i,1}\cdots p_{i,n_i-1}(1-p_{i,n_i}))$$

$$G \geq q_s \sum_{i=1}^{s-1} [\varphi_i-\varphi_s](p_1r_1\cdots p_{i-1}r_{i-1})(1-p_ir_i)$$

$$G \geq q_s[(1-p_1r_1)(\varphi_1-\varphi_s)+p_1r_1(1-p_2r_2)(\varphi_2-\varphi_s)+\cdots+p_1r_1\cdots p_{s-2}r_{s-2}(1-p_{s-1}r_{s-1}) (\varphi_{s-1}-\varphi_s)]\ ,$$

or, if we let $q_i^* = 1-p_i r_i$, so $q_i^* \geq q_i$, we have

$$(2) \quad G \geq q_s[q_1^*(\varphi_1-\varphi_s)+(1-q_1^*)q_2^*(\varphi_2-\varphi_s)+\cdots+(1-q_1^*)(1-q_2^*)\cdots(1-q_{s-2}^*)q_{s-1}^*(\varphi_{s-1}-\varphi_s)] .$$

It will be noted that expressions (1) and (2) are very similar. Both can be interpreted as q_s times the expected value of a game in which one receives reward $(\varphi_1-\varphi_s)$ with probability q_1^* (or q_1), or, if that fails the smaller reward $(\varphi_2-\varphi_s)$ with probability q_2^* (or q_2), etc. Since $q_i^* \geq q_i$, intuitively it is evident that $G \geq L$. This can be proved by differentiating L with respect to q_i, (holding φ_i constant), to obtain

$$\frac{\partial L}{\partial q_i} = (1-q_1)\ldots(1-q_{i-1})[(\varphi_i-\varphi_s)-q_{i+1}(\varphi_{i+1}-\varphi_s)-(1-q_{i+1})q_{i+2}(\varphi_{i+2}-\varphi_s) - \cdots - (1-q_{i+1})\ldots(1-q_{s-2})(q_{s-1})(\varphi_{s-1}-\varphi_s)]$$

$$\geq 0 \quad \text{since} \quad (\varphi_i-\varphi_s) \geq (\varphi_{i+1}-\varphi_s) \geq \cdots \quad \text{and}$$

$$q_{i+1}+(1-q_{i+1})q_{i+2}+\cdots+(1-q_{i+1})\ldots(1-q_{s-2})(q_{s-1}) = 1-p_{i+1}p_{i+2}\cdots p_{s-1} \leq 1 .$$

Now $G \geq L$ by the mean value theorem.

<u>Theorem 3</u>. <u>If</u> R^1 and R^2 <u>are two optimal choices of</u> r <u>obstacles</u>, <u>and if</u> r_i <u>is the best</u> $(i+1)^{st}$ <u>choice to add to</u> R^i, <u>then</u> $V(R^1 \cup \{r_1\}) = V(R^2 \cup \{r_2\})$.

<u>Proof</u>.

Case I: $R^1 \cup \{r_1\} = R^2 \cup \{r_2\}$. Then the result is trivial.

Case II: $R^1 \cup \{r_1\} \neq R^2 \cup \{r_2\}$. Then let $\bar{r}_1$ be that member of $(R^1 \cup \{r_1\})(R^2 \cup \{r_2\})^c$ with the highest φ (or one of them if more than one exist). Then

$$V(R^1 \cup \{r_1\}) = V(\{\bar{r}_1\} \cup [R^1 \cup \{r_1\} - \{\bar{r}_1\}])$$

which, by the proof of theorem two is less than or equal to $V(\{\bar{r}_1\} \cup R^2)$, which in turn is less than or equal to $V(\{r_2\} \cup R^2)$. Similarly

$$V(R^2 \cup \{r_2\}) \leq V(\{r_1\} \cup R^1) \quad \text{so} \quad V(R^2 \cup \{r_2\}) = V(R^1 \cup \{r_1\}) .$$

<u>Remark:</u> Theorem 3 implies that every optimal choice of r obstacles is contained in an optimal choice of $r+1$ obstacles.

<u>Discussion</u>. The neatness of theorem 1 compared to theorem 2 raises the question of whether theorem 2 can be strengthened. In particular, is there a choice function ψ_i, analogous to φ_i, indicating which obstacles present the most inviting targets?

Consider, for example, the following two obstacles A and B:

$$p_A = .2 \qquad\qquad p_B = .5$$
$$v_A = 3 \qquad\qquad v_B = 1 .$$

If they were combined with obstacle C, where $p_C = .5$, $v_C = 2$ and the two best are chosen, obstacle C, with the highest pv, must be in the optimal solution (by theorem 2 with r=1) and

$$V(C;A) = p_C v_C + p_C p_A v_A = (.5)(2)+(.5)(.2)(3)=1.0 + 0.3 = 1.3$$
$$V(C;B) = p_C v_C + p_C p_B v_B = (.5)(2)+(.5)(.5)(1)=1.0 + 0.25 = 1.25 .$$

Hence obstacle A is preferable to obstacle B when combined with C.

Suppose, on the other hand, that obstacles A and B are to be combined with obstacle D, with $p_D = .1$ and $v_D = 7$. We again have D in the optimal solution for two by theorem 2, and

$$V(B;D) = p_B v_B + p_B p_D v_D = (.5)(1)+(.5)(.1)(7) = (.5)+(.35) = .85$$

$$V(D;A) = p_D v_D + p_D p_A v_A = (.1)(7)+(.1)(.2)(3) = .7 + .06 = .76 .$$

Thus B is preferable to A when combined with D. By way of distinction, the optimal ordering is always C, B, D, A, according to φ, no matter which of them is chosen. Hence, contrary to the situation with φ, it turns out that ψ is not independent of the other possible choices.

However, it is true that ψ depends only on obstacles previously chosen, as theorem 2 shows. Thus if we were to choose two from $\{A,B,C,E\}$ and E were not chosen, then C and A would be.

The best we can do for ψ, then, is to give computational methods of finding the optimal r in a given instance. Three such procedures are discussed in the next section.

3. Three optimal methods.

The set of subsets of n obstacles is trivially a lattice under the inclusion relation, with $\binom{n}{r}$ nodes on the r^{th} level, $0 \leq r \leq n$, where $\binom{n}{0} = 1$. Theorem 2 says that there is at least one optimal chain and no descending optimal chains which stop, and theorem 3 ensures that there are no ascending optimal chains which stop.

Procedure I begins at the bottom of this lattice, picking the best single obstacle, the best pair, etc., thus moving up an optimal chain. Procedure II starts at the top, expelling the best profitable one at each stage as it descends an optimal chain. Procedure III starts at the appropriate level with any r obstacles, and moves across the lattice, changing one obstacle at a time, until the optimum is reached. A fuller description, an upper bound on the number of iterations required, and a proof that each reaches the optimal choice is given below:

Procedure I: (Start at the bottom).

1. Description: Pick the best one (highest $p_i v_i$), best second one to go with it, etc. However, it is not necessary to recompute the gain from adding each obstacle each time. In going from an optimal r to an optimal $r+1$, the ordering of the optimal r divides the unchosen obstacles into $r+1$ groups, where any member of the i^{th} group would be ordered i^{th} were it chosen to be in the solution for $r+1$. Thus all those obstacles in group 1 have φ greater than or equal to that of the first obstacle in the optimal r, etc.

The gain from adding any obstacle i^* in group i is

$$\begin{aligned} &V(1;2;\ldots;i-1;i^*;i;\ldots;r)-V(1;2;\ldots;i-1;i;\ldots;r) = \\ (3)\qquad &p_1v_1+\cdots+p_1p_2\cdots p_{i-1}p_{i^*}v_{i^*}+p_1p_2\cdots p_{i-1}p_{i^*}V(i;i+1;\ldots;r) \\ &-[p_1v_1+\cdots+p_1\cdots p_{i-1}v_{i-1}+p_1\cdots p_{i-1}V(i;i+1;\ldots;r)] = \\ &(p_1\cdots p_{i-1})[p_i^*v_i^*-q_i^*\ V(i;i+1;\ldots;r)] \ . \end{aligned}$$

This gives in explicit terms an expression for ψ in the i^{th} group. If a new obstacle is added, for an optimal $r+1$, which has φ greater than the $i-1^{st}$ in the optimal r, groups $i,i+1,\ldots,r+1$ will not change their ψ ordering. Thus only those groups with φ greater than or equal to the new one need be recomputed. The gain $(p_1\cdots p_{i-1})[p_i^*v_i^*-q_i^*\ V(i;\ldots;r)]$ must be recomputed, but this is easy.

2. Bound: An upper bound on the number of comparisons required is given by

$$n+(n-1)+(n-2)+\cdots+(n-r+1) = \sum_{i=1}^{n} i - \sum_{i=1}^{n-r} i = \frac{n(n+1)}{2} - \frac{(n-r)(n-r+1)}{2} = \frac{r}{2}(2n-r+1)$$

3. The procedure cannot terminate before the optimal r is reached, as follows directly from theorems two and three.

Procedure II: (Start at the top).

1. Description: Start with all n. Expell the least profitable one, then the next least profitable one, etc. The loss in expected value for removing obstacle i^* is the same as the gain in adding it, and hence is given by equation (3). The same recomputation considerations apply.

2. Bound: For Procedure II, the upper bound is given by

$$n+(n-1)+\cdots+n-(n-r)+1 = n+(n-1)+\cdots+(r+1) = \sum_{i=1}^{n} i - \sum_{i=1}^{r} i = \frac{n(n+1)}{2} - \frac{r(r+1)}{2}$$
$$= \frac{(n-r)}{2}(n+r+1) \quad .$$

3. Termination at optimal r: Follows directly, again, from theorems two and three.

Procedure III: (Start in the middle).

1. Description: IIIa (step-up, step-down). Start with any set of r obstacles (preferably a carefully chosen r) and try to add each of the $n-r$ others, one by one. After adding each, yielding a set of $r+1$ obstacles, see which one is least valuable. If the one you added is least valuable each time, then stop; if not, expell the obstacle which is least valuable, yielding a set of r obstacles with greater expected value then before and continue.

IIIb (step-down, step-up). Start with any r obstacle, delete each one in sequence, yielding $r-1$ obstacles. Now see which among the $n-r+1$

not there is most valuable to add. If it is the one you deleted each time, then stop; if not, take the most valuable one and continue.

2. Bound: $\binom{n}{r}$. But this bound will be far larger, in general, than the actual number of iterations required. The bound will be reached only if the initial solution is the worst one, then the second worst,..., up to the $\binom{n}{r}$ worst, which is best. Any prior information, such as a near optimal solution, can be used in Procedure III. In the absence of prior information, the author recommends a convex combination of pv and φ be used as a criterion for deciding which shall be the initial r, and what order in which to try new candidates.

3. Termination at the optimal r: see Theorem four.

Theorem 4. Procedure III stops only when the optimal r has been reached.

Proof. Let S be a set of r obstacles for which Procedure III terminates, i.e., there are no obstacles $a \in S$ and $b \notin S$ such that $V(S) < V(S \cup \{b\} - \{a\})$. Note that this condition suffices for both Procedure IIIa and IIIb.

S must contain an optimal first obstacle, with highest pv, as otherwise it could be substituted in last place, increasing expected value.

Suppose S contains K, an optimal set of k obstacles, but no $K^* \supset K$, K^* an optimal set of $k+1$ obstacles (for $k+1 < r$), which we know to exist by theorems two and three. Let a be that member of $S \cap K^c$ with the smallest φ, and suppose it is in position s in S. Then the obstacles in positions $s+1,\ldots,r$ are in K, and we will call the set of obstacles K_1. Let $K_2 = K \cap K_1^c$. Leaving the obstacles of

K_1 fixed, move, at some loss L of expected value, the members of K_2 (in φ order) to positions $r-k, r-k+1, \ldots, s-1$. Now substitute for $K \cup \{a\}$ the optimal $r+1$ containing K, namely $K^* = K \cup \{b\}$ for some $b \notin S$, strictly increasing the expected value. Suppose that b has position t after this substitution. Also let $S^* = S - \{a\} \cup \{b\}$.

Case I: $t \geq s$. Then the obstacles in K_2 can be moved back to their original positions in φ order at a gain G exactly equal to L, since the only permutation and change of elements has been in positions $s, s+1, \ldots, r$, thus not affecting $(S-\{a\}) \cap K_1^c = (S^*-\{b\}) \cap K_1^c$.

Case II: $t < s$. Then obstacle b divides K_2 into two separate parts. Let K_2^* be those members of K_2 with φ greater than or equal to that of b, and let $K_2' = K_2 - K_2^*$. Move, at some loss L', obstacle b to position s. Here

$$L' = \Big(\prod_{i \in [(S^* \cap K^{*c}) \cup K_2^*]} p_i\Big)\Big(\sum_{j \in K_2'} \Big(\prod_{k < j, k \in K_2'} p_k\Big) q_j q_b [\varphi_j - \varphi_b]\Big) .$$

Now K_2 can be redistributed in $S^* \cap K^{*c}$ according to its optimal order, at a gain G, again exactly equal to L. Now if obstacle b is moved back to its proper position in φ order, it will have to be moved past each of the members of K_2'. Then the gain G' from doing this will be a sum of terms including some like $q_j q_b [\varphi_j - \varphi_b]$ for each $j \in K_2'$. The coefficient in front will be a product of p's, including those in K_2' of higher or equal φ to that of j, all those in K_2^*, and perhaps some in $S^* \cap K^{*c}$. In any case there will be no p's in the coefficient which did not appear in L', so term-by-term, $G' \geq L'$.

If we let $V(A;B;C)$ be the expected value if set A is attempted first, with A in optimal order, then set B in optimal order, and then set C in optimal order, we can express the argument above for Case II as

$$\begin{aligned} V(S) &= V(S \cap [K \cup \{a\}]^c) \cup \{a\} \cup K) = V(S \cap [K \cup \{a\}]^c; K_2;\ \{a\}; K_1) + L \\ &< V(S \cap (K \cup \{a\})^c;\ K \cup \{b\}) + L = V(S^*\text{-}K^*; K_2^*; \{b\}; K_2'; K_1) + L \\ &= V(S^*\text{-}K; K_2^*; K_2'; \{b\}; K_1) + L + L' \\ &= V(S^*\text{-}K^*; K_2; \{b\}; K_1) + L + L' \\ &= V((S\text{-}K^*) \cup K_2; \{b\}; K_1) - G + L + L' \qquad (G = L) \\ &= V((S^*\text{-}K^*) \cup K_2 \cup \{b\}; K_1) - G - G' + L + L' \le V(S^*) \qquad (G' \ge L') \end{aligned}$$

so $V(S) < V(S^*)$ which is a contradiction. Hence S must contain K^*, the optimal $k+1$. Hence, by induction, S contains an optimal r.

Example of the three Procedures.

Consider the three obstacles given below:

	p	v	pv	φ	
A	.85	1	.850	5.67	V(A;B) = 1.564
B	.42	2	.840	1.45	V(C;A) = 1.5225
C	.87	.9	.783	6.02	V(C;B) = 1.5138 .

Procedure I:

Choose the first obstacle according to highest pv, namely A. Then the gain from choosing B (which is in group 2 because were it chosen to be second with A it would be second in order) is $p_A(p_B v_B) = .714$ the gain from choosing C, which is in group 1, is $p_C v_C - q_C\ V(A) = .6725$. Hence B is chosen to be second, and we thus derive the optimal pair.

Procedure II:

In this procedure we start at the top with all three chosen, and see which obstacle would cost us least to expell.

Cost of expelling

$$A = p_C(p_A v_A - q_A\ V(B)) = .62988$$

$$B = p_C p_A\ (p_B v_B) = .62118$$

$$C = p_C v_C - q_C\ V(A;B) = .57968\ .$$

Hence obstacle C costs us least to expell, and we are left with $\{A,B\}$. The resultant expected value is $V(C;A;B) - .57968 = 1.564$ as we obtained before.

Procedure III:

Suppose we use the function $f = \frac{1}{2}(pv) + \frac{1}{2}\varphi$ as the discriminator function. Then $f(A) = 3.26$, $f(B) = 1.145$, $f(C) = 3.4015$. Hence A and C would be chosen to be the initial solution. In Procedure IIIa, we would test to see whether we wanted to add B by going through exactly the same analysis as we did for Procedure II. In Procedure IIIb we would test whether we want to expell A and add B (we wouldn't), then test whether we want to expell C and add B (we would).

Discussion: Procedure I is clearly most useful for r/n small, and Procedure II for r/n large. The chief question is the usefulness of Procedure III. The advantage of Procedure III is that it can use knowledge of any near-optimal solution. The author conjectures that with a suitable convex combination of pv and φ, Procedure III will be fastest in a broad middle range of r/n for large n.

4. Conclusion.

Three procedures, all of which give the best choice of r obstacles, have been described. Once the best r have been found, we can find the distribution of the corresponding outcome, which is a random variable.

Suppose that $V(1;2;\ldots;r)$ is optimal. Then at least v_1 will be achieved with probability p_1, at least $v_1 + v_2$ with probability p_1p_2 etc. Alternatively, exactly v_1 will be achieved with probability $p_1(1-p_2)$, exactly $v_1 + v_2$ with probability $p_1p_2(1-p_3),\ldots,$ exactly $\sum_{i=1}^{r} v_i$ with probability $\prod_{i=1}^{r} p_i$. We have chosen $(1,2,\ldots r)$ to maximize the expectation of this random variable. Its variance is

$$v_1^2p_1(1-p_2)+(v_1+v_2)^2(p_1p_2(1-p_3))+\cdots+(\sum_{i=1}^{r} v_i)^2(p_1p_2\cdots p_n)-V^2(1;2;\ldots;r) .$$

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